Étale cohomology of schemes and analytic spaces

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0 Introduction

This text corresponds essentially to the four hour mini-course that was given by the author in July 2010 at the university Paris 6 (Jussieu) during the *Summer school on Berkovich spaces* which was organized by the ANR Project *Espaces de Berkovich*.

The goal of those talks was to make a survey on scheme-theoretic and Berkovich-analytic étale cohomology theories for people coming from various mathematical areas (including model theory, complex dynamics, tropical geometry...) and not necessarily familiar with algebraic geometry *à la* Grothendieck. That's the reason why we chose to rather insist on the motivation, the rough ideas behind the theory and the basic examples, than to give proofs – those are usually technically involved, and can be found in the literature. Nevertheless, the interested reader may wish to have some technical references about this subject; let me now give some of them.

About the notion of a derived functor. The theory was written down for the first time by Grothendieck in a very beautiful, and quite easily readable, paper ([4]).

About scheme-theoretic étale cohomology. Foundations are developed in [9], [10]-[12], and [14]. More precisely, [9] introduces the notion of an étale morphism, and Grothendieck's theory of the fundamental group; the three books [10]-[12] are devoted to Grothendieck topologies, general properties of the associated cohomology theories, and the étale cohomology of schemes; and the ℓ-adic formalism is introduced in [14].

But all those books, though extremely interesting and useful, are quite difficult to read because of their high degree of generality. Thus the reader may prefer have a look to [13], in which the theory is presented without (too much detailed) proofs, and with a lot of explanations about the ideas, the geometric interpretation, the analogy with usual topology and so on; he could as well read [6], which is certainly far more understandable than the SGA's, though it also requires some familiarity with schemes – but such a familiarity is however unavoidable for anybody who wants to work seriously with étale cohomology.

Let us also mention a beautiful text by Luc Illusie ([5]) about the étale cohomology from a double viewpoint, historical and mathematical.

About analytic étale cohomology. The fundamental reference for that subject is the foundational article by Berkovich himself ([1]), which is not easily readable by somebody who doesn't already master basic analytic geometry and Grothendiecks' language (Grothendieck topologies, sheaves, derived categories

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and so on). The comparison theorem of [1] has been improved in [2], which is also quite technical.

1 The general motivation

As well for schemes as for Berkovich analytic spaces, étale cohomology is a cohomology theory which behaves like singular cohomology of complex analytic spaces; that is to say, it does satisfy the same general theorems (bounds on cohomological dimension, Künneth formula, Poincaré duality...), and 'takes the expected values' on several classes of algebraic varieties (projective curves, abelian varieties) or analytic spaces (polydiscs, polyannuli, analytic tori....). Moreover, as we will see, this theory also encodes in a very natural way the Galois theory of fields, which helps thinking to the latter, which is *a priori* purely algebraic, in a quite geometrical way.

In both scheme-theoretic and analytic contexts, the need for a 'singular-like' cohomology theory came from very deep *arithmetical* conjectures:

- as far as schemes are concerned, the Weil conjectures; Weil himself suggested to prove them by defining such a cohomology, which is for that reason often called a *Weil cohomology*, and by applying it to algebraic varieties defined over finite fields; it is for that purpose that Grothendieck *et. al.* developed the theory in the sixties, and Weil conjectures were eventually proven by Deligne in 1974;

-as far as analytic spaces are concerned, the Langlands program; Drinfeld and Carayol had conjectured that the local Langlands correspondence should 'live' in the étale cohomology vector spaces of some meaningful p-adic analytic moduli spaces, whose archetype is the so-called *p-adic upper half plane* $\mathbb{P}^{1,an}_{\mathbb{Q}_p} - \mathbb{P}^1(\mathbb{Q}_p)$; but at that time, the étale cohomology of a p-adic analytic space was just not defined. It was done for the first time by Berkovich, who used it to prove a conjecture by Deligne on vanishing cycles; after that, it was shown to have the expected applications to the local Langlands program (works by Harris, Taylor, Boyer, Hausberger, Dat....).

The p-adic upper half-plane. Remind that Berkovich analytic spaces are defined over *any* (i.e., possibly Archimedean) complete valued field. In particular, $\mathbb{A}^{1,an}_{\mathbb{R}}$ does make sense; as a topological space, it is the set of multiplicative semi-norms on $\mathbb{R}[T]$ which extend the usual absolute value on $\mathbb{R}$. The map $z \mapsto (P \mapsto |P(z)|)$ is easily seen to induce an homeomorphism, and even an isomorphism of locally ringed spaces, between the quotient of $\mathbb{C}$ by the complex conjugation and $\mathbb{A}^{1,an}_{\mathbb{R}}$.

The space $\mathbb{P}^{1,an}_{\mathbb{R}}$ is obtained by glueing two copies of $\mathbb{A}^{1,an}_{\mathbb{R}}$ in the usual way; hence there is a canonical isomorphism between $\mathbb{P}^{1,an}_{\mathbb{R}}$ and the quotient of $\mathbb{P}^1(\mathbb{C})$ by the complex conjugation; therefore, $(\mathbb{P}^{1,an}_{\mathbb{R}} - \mathbb{P}^1(\mathbb{R})) \simeq \mathbb{H}$, where $\mathbb{H}$ denotes the usual upper half-plane.

This is one of the numerous reasons why $\mathbb{P}^{1,an}_{\mathbb{Q}_p} - \mathbb{P}^1(\mathbb{Q}_p)$ should be thought as the p-adic analog of $\mathbb{H}$. Note that it is a well-defined Berkovich space: indeed, as $\mathbb{Q}_p$ is locally compact, $\mathbb{P}^1(\mathbb{Q}_p)$ is compact; as a consequence, $\mathbb{P}^{1,an}_{\mathbb{Q}_p} - \mathbb{P}^1(\mathbb{Q}_p)$ is an open

subset of $\mathbb{P}^(\mathbb{Q}_p)$ (one checks that it is dense, and connected); thus it inherits a natural structure of a p-adic analytic space.

Let us make now two remarks.

First remark. In both scheme-theoretic and analytic settings, there are natural cohomology theories, namely the ones associated with the *topological spaces* involved. But though they are highly interesting, they definitely can *not* play the expected role. Let us give one reason for that : if X is a d-dimensional variety (resp. a d-dimensional paracompact analytic space) over an algebraically closed field, then its topological cohomology vanishes in rank $\geqslant d + 1$, while non-zero cohomology groups should be allowed to occur up to rank $2d$ in a singular-like cohomology theory.

Second remark (due to Serre). Let us explain why it is not possible to hope for a singular-like cohomology theory for schemes providing $\mathbb{Q}$-vector spaces (the same argument would also work *mutatis mutandis* for analytic spaces). Assume that there exists such a theory $\mathrm{H}^\bullet$, let us fix a prime number p and let $\overline{\mathbb{F}_p}$ be an algebraic closure of $\mathbb{F}_p$. There exists an elliptic curve X over $\overline{\mathbb{F}_p}$ with the following property: *the ring* $(\mathsf{End}\,X) \otimes_\mathbb{Z} \mathbb{Q}$ *is a non-split $\mathbb{Q}$-quaternion algebra, that is, a non-commutative field with center $\mathbb{Q}$ and whose dimension over $\mathbb{Q}$ is equal to 4*[1]. By our assumptions on $\mathrm{H}^\bullet$, the group $\mathrm{H}^1(X)$ is a 2-dimensional $\mathbb{Q}$-vector space, on which $\mathsf{End}\,X$ acts. The corresponding homomorphism

$$\mathsf{End}\,X \to \mathsf{End}\,\mathrm{H}^1(X) \simeq \mathsf{M}_2(\mathbb{Q})$$

induces an arrow

$$(\mathsf{End}\,X) \otimes_\mathbb{Z} \mathbb{Q} \to \ \mathsf{M}_2(\mathbb{Q}).$$

As the left-hand side ring is a (non-commutative) field, this map is injective. By comparison of dimensions it is bijective; but $\mathsf{M}_2(\mathbb{Q})$ is not a field, contradiction.

This is the reason why étale cohomology usually deals with torsion or ℓ-adic coefficients, and (almost) never with integral or rational ones.

2 The notion of a Grothendieck topology

General references: [10] *or* [6], *far easier to read.*

Classical general topology consists in axiomatizing the notion of an open subset (or of a neighborhood); in order to develop étale cohomology, Grothendieck slightly generalized this formalism, extending it to the categorical setting, and giving rise to what one calls a *Grothendieck topology*. In that approach, one 'only' axiomatizes the notion of a *covering* – this is precisely what is needed to define sheaves and their cohomology groups.

Now let us start with a category C. We will note give here the precise definition of a category; let us simply remind that one has *objects* of C, *morphisms* between them (including the identity map of every object) and that there is a notion of composition of morphisms which satisfies the usual rules.

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[1]It can be shown that X satisfies this property if and only if it is *supersingular*, that is, if and only if its p-torsion subgroup consists in *one* $\overline{\mathbb{F}_p}$-point, with multiplicity p^2.

We will moreover assume that if $Y \to X$ and $Z \to X$ are arrows of C, the *fiber product* $Y \times_X Z$ does exist in C; let us give some explanations. One says that an object T of C, equipped with two morphisms $T \to Y$ and $T \to Z$ uch

that the diagram $\begin{array}{ccc} T & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$ commutes, is a *fiber product of Y and Z over X*

if for every object S of C, and every couple of morphisms $(S \to Y, S \to Z)$ such

that $\begin{array}{ccc} S & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$ commutes, there exists a unique morphism $S \to T$ making the

diagram

$$\begin{array}{ccc} S & & \\ & T \longrightarrow Z & \\ & \downarrow \quad\quad \downarrow & \\ & Y \longrightarrow X & \end{array}$$
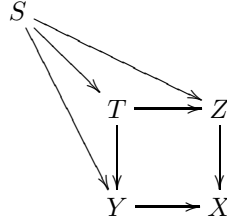
commute.

As soon as such a fiber product exists, it is unique up to canonical isomorphism; it is therefore called *the* fiber product of Y and Z over X, and denoted by $Y \times_X Z$.

Examples of categories in which $Y \times_X Z$ always exists.

• *The category of sets.* In this case $Y \times_X Z$ is the set-theoretic fiber product, that is, the set of couples (y, z) with $y \in Y$ and $z \in Z$ such that y and z have the same image on X.

• *The category of topological spaces.* In this case $Y \times_X Z$ is the set-theoretic fiber product *endowed with the product topology.*

• *The category of schemes.* Let us assume for simplicity that X, Y and Z are affine, say respectively equal to $\mathsf{Spec}\, A, \mathsf{Spec}\, B$ and $\mathsf{Spec}\, C$. Then $Y \times_X Z$ is nothing but $\mathsf{Spec}\, (B \otimes_A C)$.

• *The category of analytic spaces over a non-Archimedean, complete field.* Let us assume for simplicity that X, Y and Z are affinoid, say respectively equal to $\mathscr{M}(A), \mathscr{M}(B)$ and $\mathscr{M}(C)$. Then $Y \times_X Z$ is nothing but $\mathscr{M}(B \widehat{\otimes}_A C)$.

• *The category of the open subsets of a given topological space, in which morphisms are the inclusions.* In that case, $Y \times_X Z$ is nothing but the intersection $Y \cap Z$.

• *The category of analytic domains of a given analytic space, in which morphisms are the inclusions.* In that case, $Y \times_X Z$ is nothing but the intersection $Y \cap Z$.

Now, let us go back to our category C in which the fiber products are assumed to exist. A *Grothendieck topology* on C consists in the datum, for every object X of C, of a class of families of arrow $(X_i \to X)_i$, which are called *coverings* of X, and which are subject to the following axioms:

1) If $Y \to X$ is an isomorphism, it is a covering of X;

2) if $(X_i \to X)_i$ is a covering of X, and if $Y \to X$ is any morphism,

$$(X_i \times_X Y)_i \to Y$$

is a covering of Y;

3) if $(X_i \to X)_i$ is a covering of X and if $(X_{ij} \to X_i)_j$ is for every i a covering of X_i then $(X_{ij} \to X)_{i,j}$ is a covering of X.

For the moment, we will only give two examples, which are quite set-theoretic; we will define further the étale topology on a scheme (or an analytic space) which will be far more categorical.

First example. On the category of open subsets of a given topological space, the usual open coverings define a Grothendieck topology.

Second example. On the category of analytic domains of a given analytic space, the admissible coverings define a Grothendieck topology; it is called the *G-topology*.

Definition. A *site* is a category equipped with a Grothendieck topology.

Presheaves and sheaves

We fix from now on a site C.

Comments on the vocabulary. Though the behavior of C can be slightly different from that of an honest topological space, one will often use, while working with C, the vocabulary and, to some reasonable extent, the intuition of classical topology. Let us give a quite vague example: if X is an objet of C and if P is a property, we will say that P is true *locally on X* if there exists a covering $(X_i \to X)$ such that P is true on every X_i; we will say that P is local if it is true on a given object of C as soon as it is true locally on it.

The notion of a presheaf. A *presheaf* $\mathscr{F}$ on C is a *contravariant functor* from C to the category of sets; that is, an assignment which sends any object X of C to a set $\mathscr{F}(X)$, together with the datum, for every arrow $Y \to X$ in C, of a *restriction map* $\mathscr{F}(X) \to \mathscr{F}(Y)$ (note that it goes 'the wrong way'), all of this having to behave well with respect to the composition of arrows. The elements of $\mathscr{F}(X)$ for a given X will sometimes be called *sections of $\mathscr{F}$ on X*.

If there is no risk of confusion[2], the restriction $\mathscr{F}(X) \to \mathscr{F}(Y)$ will be denoted $s \mapsto s_{|Y}$.

A *map* from a presheaf $\mathscr{F}$ to a presheaf $\mathscr{G}$ consists in a family of maps $\mathscr{F}(X) \to \mathscr{G}(X)$, where X goes through the collection of all objects of C, which is compatible with restriction maps.

The notion of a sheaf. A presheaf $\mathscr{F}$ on C is said to be a *sheaf* if it satisfies the following condition: *for every object X of C, for every covering $(X_i \to X)$, and for every family $(s_i) \in \prod_i \mathscr{F}(X_i)$ such that*

$$s_{i|X_i \times_X X_j} = s_{j|X_i \times_X X_j} \in \mathscr{F}(X_i \times_X X_j)$$

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[2]There could be one because $\mathscr{F}(X) \to \mathscr{F}(Y)$ not only depends on Y, but on the arrow $Y \to X$.

for all (i,j), there exists a unique $s \in \mathscr{F}(X)$ such that $s_{|X_i} = s_i$ for every i.

The section s shoulds be thought as beeing obtained by *glueing* the s_i's, the equalities $s_{i|X_i \times_X X_j} = s_{j|X_i \times_X X_j}$ beeing the coincidence conditions that make the glueing process possible.

If $\mathscr{F}$ and $\mathscr{G}$ are two sheaves on C, a map from $\mathscr{F}$ to $\mathscr{G}$ is simply a map between the *presheaves* $\mathscr{F}$ and $\mathscr{G}$.

Link with old-fashioned sheaves. If C is the category of all open subsets of a given topological space T, equipped with the aforementioned Grothendieck topology, then a sheaf on C is nothing but a classical sheaf on the topological space T.

Presehaves and sheaves with some extra-structures. We have actually just defined presheaves and sheaves of *sets*. We will also use the notion of presheaves and sheaves of abelian groups. The definitions are almost the same, except that $\mathscr{F}(X)$ comes for every object X of C with the structure of an abelian group and that restrictions map are required to be homomorphisms of groups ; a morphism between two presheaves or sheaves of abelian groups $\mathscr{F}$ and $\mathscr{G}$ is a map from $\mathscr{F}$ to $\mathscr{G}$ such that $\mathscr{F}(X) \to \mathscr{G}(X)$ is a morphism of groups for every object X of C.

One can of course define in an analogous way presheaves and sheaves of rings, of k-vector spaces or k-algebras (for k a field), etc. , and morphisms between them.

Subsheaf of a sheaf. If $\mathscr{F}$ is a sheaf on C, a *subsheaf* of $\mathscr{F}$ is a sheaf $\mathscr{G}$ such that $\mathscr{G}(X)$ is for every object X of C a subset of $\mathscr{F}(X)$. If $\mathscr{G}$ is a subsheaf of $\mathscr{F}$, the property for a section of $\mathscr{F}$ to be a section of $\mathscr{G}$ is a local property.

Presheaf associated with a sheaf. Every presheaf $\mathscr{F}$ on C can be sheafified in a natural way; if A is a set and if $\mathscr{F}$ is the constant presheaf $U \mapsto A$, its sheafification will be denoted by $\underline{A}_{\mathsf{C}}$ (or $\underline{A}$, or A_C, or A if there is no risk of confusion), and called the *constant sheaf* (associated with) A. When C is the site of open subsets of a given topological space X, the sheaf $\underline{A}$ is the sheaf of locally constant A-valued functions on X.

If $\mathscr{O}$ is a sheaf of commutative rings on C and if $\mathscr{F}$ and $\mathscr{G}$ are two sheaves of $\mathscr{O}$-modules, the presheaf $U \mapsto \mathscr{F}(U) \otimes_{\mathscr{O}(U)} \mathscr{G}(U)$ is not a sheaf in general; its sheafification is denoted by $\mathscr{F} \otimes_{\mathscr{O}} \mathscr{G}$.

A crucial notion: the *sheaf theoretic* image of a map. Let $\mathscr{F}$ and $\mathscr{G}$ be two sheaves on C, and let f be a map from $\mathscr{F}$ to $\mathscr{G}$; one would like to define a subsheaf of $\mathscr{G}$ which could be reasonably thought as beeing the image of f. The problem is that the 'naive' image of f, that is, the assignment

$$X \mapsto \{t \in \mathscr{G}(X), \exists\, s \in \mathscr{F}(X) \text{ s.t. } t = f(s)\},$$

is a presheaf *which has no reason to be a sheaf.* The point is the following: for a section of $\mathscr{G}$, belonging to the naive image of f, is *not* local in general (an example is given a little bit further).

To bypass this difficulty, one defines the *sheaf-theoretic image* of the map f as the assignment that sends an object X of C to the set of sections of $\mathscr{G}$ on X which belong *locally* to the naive image of f; in more precise words, X is sent to

the set of $t \in \mathscr{G}(X)$ satisfying the following properties: *there exists a covering* (X_i) *of* X *and, for every* i, *an element* s_i *of* $\mathscr{F}(X_i)$ *such that* $f(s_i) = t_{|X_i}$ *for all* i.

This assignment turns out to be a sheaf we will denote by $\mathsf{Im}\ f$; it can be shown that this is the sheafification of the naive image of f; this is also the smallest subsheaf of $\mathscr{G}$ containing it. We will say that f is *surjective* if $\mathsf{Im}\ f = \mathscr{G}$.

A fundamental example. Let $\mathscr{O}$ be the sheaf of holomorphic functions on $\mathbb{C}$. The fact for a holomorphic function to admit a primitive on a given open subset of $\mathbb{C}$ is not a local property – for instance, $1/z$ has locally a primitive on $\mathbb{C}^*$, but not globally; hence the naive image of the derivation $d : \mathscr{O} \to \mathscr{O}$ is not a subsheaf of $\mathscr{O}$.

However, the sheaf-theroetic image $\mathsf{Im}\ d$ of d is very easy to describe: as every holomorphic function on an open subset of $\mathbb{C}$ has locally a primitive, $\mathsf{Im}\ d$ is nothing but the whole sheaf $\mathscr{O}$.

Exact sequences of sheaves. Now let $\mathscr{F}, \mathscr{F}'$ and $\mathscr{F}''$ be three sheaves of abelian groups on C, and let $f : \mathscr{F}' \to \mathscr{F}$ and $g : \mathscr{F} \to \mathscr{F}''$ be two morphisms. We will say that the sequence

$$0 \longrightarrow \mathscr{F}' \stackrel{f}{\longrightarrow} \mathscr{F} \stackrel{g}{\longrightarrow} \mathscr{F}'' \longrightarrow 0$$

is exact if the following conditions hold:

i) for every object X of C, the sequence $0 \to \mathscr{F}'(X) \to \mathscr{F}(X) \to \mathscr{F}''(X)$ is exact (in the usual sense for a sequence of abelian groups);

ii) $\mathscr{F}'' = \mathsf{Im}\ g$ (in other words: g is surjective).

Comment. Note that in i) we *do not* require $\mathscr{F}(X) \to \mathscr{F}''(X)$ to be surjective for every X; we simply demand in ii) that g is surjective, which is slightly weaker.

Example. Let $\mathscr{O}$ be the sheaf of holomorphic functions on $\mathbb{C}$ and let d be the derivation; remind that $\underline{\mathbb{C}}$ denotes the constant sheaf associated with $\mathbb{C}$, that is, the sheaf of locally constant complex-valued functions, which embeds naturally in $\mathscr{O}$. One immediatly checks that the sequence

$$0 \longrightarrow \underline{\mathbb{C}} \longrightarrow \mathscr{O} \stackrel{d}{\longrightarrow} \mathscr{O} \longrightarrow 0$$

is exact (the fact that $\mathsf{Im}\ d = \mathscr{O}$ has be mentioned above); nevertheless, note that the derivation $\mathscr{O}(\mathbb{C}^*) \to \mathscr{O}(\mathbb{C}^*)$ is *not* onto: again, consider the function $1/z$.

3 Cohomology of sheaves

General references: [11] *and* [6]*, and* [4]*for the notion of a derived functor.*

Abelian categories, left-exact functors, and derived functors

We will not give here the precise definition of an *abelian category*. Let us just say that this is a category A which 'looks like' the category of modules over a ring; as an example, in such a category:

- the set of morphisms between two given objects is an abelian group ;
- there is a zero object;
- one can define the direct sum of a family of objects;
- one can define the sub-objects of a given object, and the quotient of an object by one of its sub-objects;
- one can define the image, the kernel, and the cokernel of a map, and hence the notions of surjective and an injective maps;
- one can define exact sequences.

Moreover, all those notions behave the usual way.

Examples. Let us give two typical examples of abelian categories: the category of modules over a given ring (non necessarily commutative); the category of sheaves of abelian groups on a site.

Definition. Let $F : \mathsf{A} \to \mathsf{B}$ be a functor between two abelian categories. We say that F is *exact* (resp. *left-exact*) if for every exact sequence

$$0 \to X' \to X \to X'' \to 0$$

in A, the induced sequence

$$0 \to F(X') \to F(X) \to F(X'') \to 0 \quad (\text{resp. } 0 \to F(X') \to F(X) \to F(X''))$$

is exact.

Comment. A left-exact functor is exact if and only if it carries surjections to surjections.

Remark. There is of course also a notion of a *right-exact* functor, but we will not use it here.

Examples.

1) Let C be a site and let X be an object of C. Let A be the category of sheaves of abelian groups on C, and let B be the category of abelian groups. Let Γ be the functor from A to B that sends a sheaf $\mathscr{F}$ to the group $\mathscr{F}(X)$.

By the very definition of an exact sequence of sheaves, Γ is left-exact. It is not exact in general: take for C the site of open subsets of $\mathbb{C}$, for $\mathscr{O}$ the sheaf of holomorphic functions and for X the open subset $\mathbb{C}^*$ of $\mathbb{C}$. Let $d : \mathscr{O} \to \mathscr{O}$ be the derivation map. As we have already mentioned, d is surjective but the induced map $\mathscr{O}(\mathbb{C}^*) \to \mathscr{O}(\mathbb{C}^*)$ is not onto ($1/z$ doesn't belong to its image); hence Γ is not exact.

2) Let X be a Hausdorff topological space, let A be the category of sheaves of abelian groups on X, and let B be the category of abelian groups. Let Γ_c be the functor from A to B that sends a sheaf $\mathscr{F}$ to the subgroup $\mathscr{F}(X)$ that consists in sections with compact support. By the very definition of an exact sequence of sheaves, Γ_c is left-exact, but not exact in general.

As an example, let X be the open interval $]0; 1[$, let $\mathscr{F}$ be the sheaf of real analytic functions on X and let $\mathscr{G}$ be the sheaf on X that sends an open subset U to $\mathbb{R}$ if $\frac{1}{2} \in U$ and to 0 otherwise; evaluation at $\frac{1}{2}$ induces a natural map $\mathscr{F} \to \mathscr{G}$ which is easily seen to be surjective. Nevertheless, $\Gamma_c(\mathscr{F}) \to \Gamma_c(\mathscr{G}) = 0 \to \mathbb{R}$ is not onto; hence Γ_c is not exact.

3) Let G be a group. Let A be the category of G-modules, that is, of abelian groups endowed with an action of G (or, equivalently, of modules over the ring $\mathbb{Z}[G]$), and let B be the category of abelian groups. Let F be the functor from A to B that send a G-module M to its subgroup M^G consisting of G-invariant elements. It is an easy exercise to see that F is left-exact. Let us show by a counter-example that it is not exact in general.

Let us take for G the group $\mathbb{Z}/2\mathbb{Z}$, acting on $\mathbb{C}^*$ through complex conjugation. The morphism $\mathbb{C}^* \stackrel{z \mapsto z^2}{\longrightarrow} \mathbb{C}^*$ is equivariant and surjective; but after taking the G-invariants one gets $z \mapsto z^2$ from $\mathbb{R}^*$ to $\mathbb{R}^*$, which is not onto anymore.

4) There is a topological avatar of the preceeding example, which will play a central role in étale cohomology: let G be a *profinite group* (e.g. $G = \mathrm{Gal}\,\overline{k}/k$, for k a field and $\overline{k}$ an algebraic closure of k). Let A be the category of *discrete* G-modules, that is, of G-modules such that the stabilizer of every element is an *open* subgroup of G; it is an abelian category. Let B be the category of abelian groups, and let F be the functor from A to B that sends a G-module M to its subgroup M^G consisting of G-invariant elements. It is left-exact, but not exact in general (the above counter-example fits in that setting: as $\mathbb{Z}/2\mathbb{Z}$ is finite, it is profinite and every $\mathbb{Z}/2\mathbb{Z}$-module is discrete).

The notion of a derived functor. Derived functors of a given left-exact functor are a way to encode its lack of exactness; let us be now more precise.

Let $F : \mathsf{A} \to \mathsf{B}$ be a left-exact functor between abelian categories. A collection $(\mathrm{R}^i F)_{i \geqslant 0}$ of functors from A to B is said to be a *collection of derived functors of F* if the following hold:

• $\mathrm{R}^0 F = F$;
• to every short exact sequence $0 \to X' \to X \to X'' \to 0$ of objects of A one can associate in a functorial way a long exact sequence

$$0 \to F(X') \to F(X) \to F(X'') \to \mathrm{R}^1 F(X') \to \mathrm{R}^1 F(X) \to \mathrm{R}^1 F(X'') \to \mathrm{R}^2 F(X') \to \dots$$

$$\dots \to \mathrm{R}^i F(X') \to \mathrm{R}^i F(X) \to \mathrm{R}^i F(X'') \to \mathrm{R}^{i+1} F(X') \to \dots$$

• the collection $(\mathrm{R}^i F)$ is universal for the preceeding properties, in some straightforward sense we won't explicit.

If such a collection exists, it is unique up to a canonical system of isomorphisms of functors; hence if it is the case, we will speak of *the* collection $(\mathrm{R}^i F)$ of the derived functors of F, and we will say that $\mathrm{R}^i F$ is *the i-th derived functor of F.

A stupid example. If F is an exact functor, then it does admit derived functors: one has $\mathrm{R}^0 F = F$ and $\mathrm{R}^i F = 0$ for positive i.

Conditions for the existence of derived functors. Under a rather technical condition on the abelian category A, namely, if it has *enough injective objects*, **every left-exact functor from A to another abelian category B does admit derived functors.**

Let us now make three remarks.

1) The aforementioned condition is always fulfilled by the following categories we have already encountered: the category of sheaves of abelian groups on a

site; the category of modules over a ring, including that of G-modules for G a group; the category of *discrete* G-modules, for G a profinite group[3]. Hence on those categories, derived functors will always exist.

2) The proof of the existence of derived functors (under the above condition) proceeds by *building* them, but in a highly abstract way which is usually not helpful at all when one needs to actually compute them.

3) If G is a (profinite) group, the derived functors of $M \mapsto M^G$, with M going through the category of (discrete) G-modules, can be computed explicitly using (continuous) G-cochains, cocycles and coboundaries with coefficients in M. They will be denoted by $\mathrm{H}^\bullet(G, .)$.

About the conditions of having 'enough injective objects'. Let A be an abelian category. An object X of A is said to be *injective* if every injective arrow from X to an object of A admits a retraction or, otherwise said, identifies X with a direct summand of the target. As an example, note that in the category of vector spaces over a given field, every object is injective.

That is not the case for that of abelian groups; indeed, $\mathbb{Z}$ is not an injective object of this category: simply note that the injection $n \mapsto 2n$ from $\mathbb{Z}$ into himself doesn't admit a retraction – indeed, $2\mathbb{Z}$ is not a direct summand of $\mathbb{Z}$.

In fact, one can prove (exercise !) that an abelian group is injective if and only if it is divisible.

An abelian category A is said to have *enough injectives* if every object of A admits an injective arrow into an injective object. The category of abelian groups has enough injectives: it is not difficult (again, exercise !) to embed a given abelian group in a divisible one ($\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an example of such an embedding). This fact plays a crucial role in the proof that the category of modules over a given ring and the category of sheaves of abelian groups on a site have enough injectives.

The cohomology groups of a sheaf

Let us now fix a site C. Let X be an object of C; let A be the category of sheaves of abelian groups on C and let B be the category of abelian groups. As we have seen, the functor from A to B that sends $\mathscr{F}$ to $\mathscr{F}(X)$ is left-exact; it thus admits derived functors (see above).

Notation and definition. The functor $\mathscr{F} \mapsto \mathscr{F}(X)$ will also be denoted $\mathscr{F} \mapsto \mathrm{H}^0(X, \mathscr{F})$. For very $i > 0$, its i'th derived functor will be denoted by $\mathscr{F} \mapsto \mathrm{H}^i(X, \mathscr{F})$. We will say that $\mathrm{H}^i(X, \mathscr{F})$ is *the i-th cohomology group of X with coefficients $\mathscr{F}$*, or *the i-th cohomology group of $\mathscr{F}$ on X*.

Links with other cohomology theories. Let us assume that C is the site of open subsets of a given topological space X. For every abelian group A, remind that $\underline{A}$ denotes the sheaf of locally constant A-valued functions on X.

Then if X is a 'reasonable' topological space (e.g. a metrizable locally finite CW-complex) it can be shown that for every i the group $\mathrm{H}^i(X, \underline{A})$ in the above sense does coincide with the i-th group of *singular* cohomology of X with coefficients A.

If Y is a closed subset of X (resp. if X is Hausdorff), one can also derive the functor of global sections with support in Y (resp. compact support). The

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[3]The latter category is in fact (equivalent to) the category of sheaves of abelian groups on a suitable site.

resulting groups are denoted by $\mathrm{H}_Y^\bullet(X,.)$ (resp. $\mathrm{H}_c^\bullet(X,.)$) and called the *cohomology groups with support in Y (resp. with compact supports)* of the sheaf involved. Again, under reasonable assumptions on X and Y (resp. on X) the group $\mathrm{H}_Y^i(X,\underline{A})$ (resp. $\mathrm{H}_c^i(X,\underline{A})$) in that sense coincide with the i-th group of *singular* cohomology of X with support in Y (resp. with compact support) and with coefficients A.

Some words about the restriction. Let us still assume that C is the site of open subsets of a topological space X, let U be an open subset of X and let V be an open subset of U; let $\mathscr{F}$ be a sheaf of abelian groups on C. It defines by restriction a sheaf $\mathscr{F}_{|U}$ on the topological space U.

Now one can define for every i, in the above sense, the groups $\mathrm{H}^i(V,\mathscr{F})$ and $\mathrm{H}^i(V,\mathscr{F}_{|U})$: for the first one, one computes the i-th derived functor of $\mathscr{G} \mapsto \mathscr{G}(V)$ for $\mathscr{G}$ going through the category of abelian sheaves *on X*, and one applies it to $\mathscr{F}$; for the second one, one computes the i-th derived functor of $\mathscr{G} \mapsto \mathscr{G}(V)$ for $\mathscr{G}$ going through the category of abelian sheaves *on U*, and one applies it to $\mathscr{F}_{|U}$.

But it turns out that there is no reason to worry about it: both groups $\mathrm{H}^i(V,\mathscr{F})$ and $\mathrm{H}^i(V,\mathscr{F}_{|U})$ are canonically isomorphic; hence we will most of the time only use the notation $\mathrm{H}^i(V,\mathscr{F})$.

About the restriction in a more general setting. The phenomenon we have just mentioned extends to every site C, as follows.

If X is an object of C, let C/X be the so-called category *of objects of C over X*, which is defined in the following way:

• its objects are couples (Y,f) where $Y \in \mathsf{C}$ and where f is an arrow from Y to X;
• if (Y,f) and (Z,g) are two objects of C/X, an arrow from (Y,f) to (Z,g) is nothing but a morphism $h : Y \to Z$ such that $g \circ h = f$.

The category C/X inherits a Grothendieck topology: if (Y,f) is an object of C/X, a family $((Y_i,f_i) \to (Y,f))$ of arrows in C/X is said to be a covering if and only if $(Y_i \to Y)$ is a covering. If $\mathscr{F}$ is a sheaf on C, the assignment $(Y,f) \mapsto \mathscr{F}(Y)$ defines a sheaf on C/X, which is denoted by $\mathscr{F}_{|X}$.

Now if $(Y,f) \in \mathsf{C}/X$, one can define for every integer i the groups $\mathrm{H}^i(Y,\mathscr{F})$ and $\mathrm{H}^i((Y,f),\mathscr{F}_{|X})$: for the first one, one computes the i-th derived functor of $\mathscr{G} \mapsto \mathscr{G}(Y)$ for $\mathscr{G}$ going through the category of abelian sheaves *on C*, and one applies it to $\mathscr{F}$; for the second one, one computes the i-th derived functor of $\mathscr{G} \mapsto \mathscr{G}(Y,f)$ for $\mathscr{G}$ going through the category of abelian sheaves *on C/X*, and one applies it to $\mathscr{F}_{|X}$.

But it turns out that there is no reason to worry about it: both groups $\mathrm{H}^i(Y,\mathscr{F})$ and $\mathrm{H}^i((Y,f),\mathscr{F}_{|X})$ are canonically isomorphic; hence we will most of the time only use the notation $\mathrm{H}^i(Y,\mathscr{F})$.

Cup-products. If $\mathscr{F}$ and $\mathscr{G}$ are two sheaves of abelian groups on a site C, one has for every (i,j) and every $X \in \mathsf{C}$ a natural *cup-product pairing*

$$\cup : \mathrm{H}^i(X,\mathscr{F}) \otimes \mathrm{H}^j(X,\mathscr{G}) \to \mathrm{H}^{i+j}(X,\mathscr{F} \otimes_{\underline{\mathbb{Z}}} \mathscr{G}).$$

It is 'graded-commutative' : if $h \in \mathrm{H}^i(X,\mathscr{F})$ and $h' \in \mathrm{H}^j(X,\mathscr{G})$ then one has $h' \cup h = (-1)^{ij} h \cup h'$.

What is étale cohomology ?

In both scheme-theoretic and Berkovich's settings, one will define the notion of an *étale map*.

Now, for X being a scheme (resp. a Berkovich analytic space) we will consider the site $X_{\text{ét}}$ defined as follows:

- its objects are couples (U, f) where U is a scheme (resp. an analytic space) and $f : U \to X$ an étale map (though we often will simply denote such an object by U, one has to keep in mind that f is part of the data);
- a morphism from (V, g) to (U, f) is a map $h : U \to V$ such that $f \circ h = g$; *such a map is automatically itself étale.*
- a family $(g_i : U_i \to U)$ is a covering if and only if it is *set-theoretically* a covering, that is, if and only if $U = \bigcup g_i(U_i)$.

We will call $X_{\text{ét}}$ the *étale site of X*, and its topology will be the *étale topology* on X; an *étale sheaf on X* will be a sheaf on $X_{\text{ét}}$.

The *étale cohomology* will then simply denote the cohomology theory of étale sheaves.

4 Étale morphisms of schemes

General reference: [9] or [6].

Motivation, definition and first properties

Let us begin by some vocabulary. If Y and X are two topological spaces (resp. two complex analytic spaces) and if $y \in Y$ we will say that a continuous (resp. holomorphic) map $f : Y \to X$ is *a local homemorphism at y* (resp. *a local isomorphism at y*) if there exists an open neighborhood V of y and an open neighborhood U of $f(y)$ such that f induces an homeomorphism (resp. a holomorphic diffeomorphism) $V \simeq U$.

What is an étale map ? The rough idea one should keep in mind is the following: *the fact for a morphism of schemes to be étale at a point of the source is an algebraic analog of the fact for a continuous map between topological spaces to be a local homeomorphism at a point of the source.*

Before giving a precise definition, let us give an example of a local homeomorphism coming from differential geometry which we will essentially mimic in the scheme-theoretic context. Let X be a C^∞-manifold, let n be an integer and let $P_1, \ldots, P_n$ be polynomials in n variables $T_1, \ldots, T_n$ with coefficients in $C^\infty(X)$; let us denote by Y the subset of $X \times \mathbb{R}^n$ which consists in points $(x, t_1, \ldots, t_n)$ such that $P_i(x, t_1, \ldots, t_n) = 0$ for every i, and by $f : Y \to X$ the map induced by the first projection.

Let y be a point of Y at which the Jacobian determinant $\left| \dfrac{\partial P_i}{\partial T_j} \right|_{i,j}$ doesn't vanish; it follows from the implicit function theorem that f is a local homeomorphism at y.

Definition. Let $\varphi : Y \to X$ be a morphism of schemes, let y be a point of Y and let x be its image. We will say that φ is *étale* at y if there exists an

affine neighborhood $U = \mathsf{Spec}\ A$ of x in X, a neighborhood V of y in $f^{-1}(U)$, polynomials $P_1, \ldots, P_n$ in $A[T_1, \ldots, T_n]$, and a commutative diagram

$$
\begin{array}{ccc}
& & \mathsf{Spec}\ A[T_1, \ldots, T_n]/(P_1, \ldots, P_n) \\
& \nearrow^{\psi} & \downarrow \\
V & \longrightarrow & \mathsf{Spec}\ A
\end{array}
$$

with ψ an open immersion such that $\left| \dfrac{\partial P_i}{\partial T_j} \right|_{i,j}$ doesn't vanish on $\psi(V)$.

We will say that φ is *étale* if it is étale at every point of Y.

Some comments.

• It follows from the definition that the étale locus of a given map is an open subset of its source.

• If $\varphi : Y \to X$ is a morphism which is étale at some point y of Y, one can always find a local presentation of φ as above *with $n = 1$* (and then one has only to ensure that the partial derivative $\partial P / \partial T$ doesn't vanish on $\psi(V)$).

• There is another definition of étaleness, which is more technically involved: a map $Y \to X$ is étale at y if and only if it is *flat* and *unramified* at y.

• We are now going to give some basic properties of étale maps; for the sake of simplicity, we restrict ourselves to *global* étale morphisms; but most of those results also have a local counterpart.

First properties of étale maps. Every étale map is open. If $Z \to Y$ and $Y \to X$ are étale, then the composite map $Z \to X$ is étale too. If $Y \to X$ is étale and if $X' \to X$ is *any* morphism, then $Y \times_X X' \to X'$ is étale. If $Y \to X$ and $Z \to X$ are étale, then *any* morphism $Y \to Z$ making the diagram

$$
\begin{array}{ccc}
Y & \longrightarrow & Z \\
& \searrow & \downarrow \\
& & X
\end{array}
$$

commute is étale.

If you find that the latter property sounds a bit strange, think to the analogous fact for local homeomorphisms in topology – which is obvious.

Examples of étale morphisms.

1) Every open immersion is étale.

2) If $Y \to X$ is a morphism between two complex algebraic varieties (*i.e* between two schemes of finite type over $\mathbb{C}$) then it is étale if and only if the induced holomorphic map $Y(\mathbb{C}) \to X(\mathbb{C})$ is a local isomorphism on $Y(\mathbb{C})$.

3) We are now going to give three fundamental examples of *finite* étale maps (such a map is also called an étale cover – this is the analog of the notion of a topological cover, see further the discussion about the étale fundamental group).

- Let A be a commutative ring, let $f \in A^*$, and let n be an integer which is invertible in A. The finite morphism

$$\mathsf{Spec}\, A[T]/(T^n - f) \to \mathsf{Spec}\, A$$

is étale; this follows from the fact that $\dfrac{\partial(T^n - f)}{\partial T}$ doesn't vanish on $\mathsf{Spec}\, A[T]/(T^n - f)$. Indeed, for every point x of the latter one has $T^n(x) = f(x) \neq 0$, whence $nT^{n-1}(x) \neq 0$ (we have used the fact that both f and n are invertible in A).

Note that in the particular case where $A = k[X, X^{-1}]$ for k a field in which $n \neq 0$, and where $f = X$, the above finite étale map is nothing but $\mathbb{G}_m \xrightarrow{x \mapsto x^n} \mathbb{G}_m$.

- Let p be a prime number, let A be an $\mathbb{F}_p$-algebra and let $f \in A$. The finite morphism

$$\mathsf{Spec}\, A[T]/(T^p - T - f) \to \mathsf{Spec}\, A$$

is étale; this follows from the fact that $\dfrac{\partial(T^p - T - f)}{\partial T} = 1$, hence is invertible on $\mathsf{Spec}\, A[T]/(T^p - T - f)$.

Note that in the particular case where $A = k[X]$ for k a field of char. p, and where $f = X$, the above finite étale map is nothing but $\mathbb{A}^1_k \xrightarrow{x \mapsto x^p - x} \mathbb{A}^1_k$. Hence it appears as a *non-trivial étale cover of the affine line* (of degree p, by itself). Note that this example is 'geometric' and not 'arithmetic' in the sense that it holds for *every* field of char. p, even algebraically closed.

We thus see that *the affine line over an algebraically closed field of char. p is never simply connected*.

- Let k be a field. A finite extension L of k is said to be *separable* if it is isomorphic to $k[X]/P$ for some irreducible $P \in k[X]$ with $P' \neq 0$ (this is equivalent to the fact that $(P, P') = 1$). If char. $k = 0$, if k is algebraically closed or if k is finite, every finite extension of k is separable. A finite k-algebra is said to be separable if it is a product of finitely many finite separable extensions of k.
One can prove that if X is a scheme of finite type over k, it is étale over k if and only if it is isomorphic to the spectrum of a finite separable k-algebra (and X is then finite étale over k).
Note that if k is algebraically closed, a finite separable k-algebra is an algebra isomorphic to k^n for some n ; hence a k-scheme of finite type is étale over k if and only if it is a disjoint sum of finitely many copies of $\mathsf{Spec}\, k$.
The latter assertion is a very simple illustration of the following principle: from the étale viewpoint, the spectrum of an algebraically (and, more generally, separably) closed field is analog to a one-point space in topology; this is definitively false for the spectrum of an arbitrary field: for example, as $\mathbb{C}$ is a separable extension of $\mathbb{R}$, the map $\mathsf{Spec}\, \mathbb{C} \to \mathsf{Spec}\, \mathbb{R}$ is a *non-trivial étale cover of degree 2* of $\mathsf{Spec}\, \mathbb{R}$.

The fibers of an étale map. Let $\varphi : Y \to X$ be an étale morphism of schemes, and let $x \in X$. The fiber $\varphi^{-1}(x)$ is discrete, and finite soon as φ is of finite type (*e.g.* Y and X are two algebraic varieties over a given field); scheme-theoretically one can write $\varphi^{-1}(x) = \coprod \operatorname{\mathsf{Spec}} k_i$, with each k_i finite separable over $\kappa(x)$.

The étale topology on a scheme.

Let X be a scheme. We have defined what an étale morphism of schemes is. This allows us to define the étale site $X_{\text{ét}}$ and to speak about the étale topology on X (see just before section 4); we now want to give an example of a property which is satisfied locally for the étale topology, but not in general for the Zariski one.

Let $Y \to X$ be a morphism of schemes, and $y \in Y$. We will say that $Y \to X$ is *smooth at y* if there exists an integer n, an open neighborhood U of y and a factorization of $U \to X$ through an étale map $U \to \mathbb{A}^n_X$; note that if $Y \to X$ is étale at y, it is smooth at y (take $n = 0$ and U beeing the étale locus of $Y \to X$).

Now let k be a field and let X be a non-empty, smooth k-scheme. By definition of smoothness, there exists a non-empty open subset U of X and étale map $U \to \mathbb{A}^n_k$ for some n. As $U \to \mathbb{A}^n_k$ is étale, its image is a non-empty open subset V of $\mathbb{A}^n_k$.

Let k^s be a separable closure of k; it is an infinite field, therefore V admits a k^s-point; but this exactly means that there exists $x \in V$ such that $\kappa(x)$ embeds into k^s, hence is finite separable over k.

By the very definition of V, there exists $y \in U$ whose image on $\mathbb{A}^n_k$ is x. As $V \to U$ is étale, $\kappa(y)$ is finite separable over $\kappa(x)$, hence over k.

We therefore have shown that X has a point whose residue field is separable over k ; let us rephrase this property in two different ways which are tautologically equivalent (it follows from the universal property of the fiber product).

First way. There exists a finite separable extension F of k and a commutative diagram

$$
\begin{array}{ccc}
 & & X \\
 & \nearrow & \downarrow \\
\operatorname{\mathsf{Spec}} F & \longrightarrow & \operatorname{\mathsf{Spec}} k
\end{array}
$$

Second way. There exists a finite separable extension F of k and a section of the morphism $(X \times_{\operatorname{\mathsf{Spec}} k} \operatorname{\mathsf{Spec}} F) \to \operatorname{\mathsf{Spec}} F$.

For very finite separable extension F of k, the map $\operatorname{\mathsf{Spec}} F \to \operatorname{\mathsf{Spec}} k$ is étale; as it is trivially surjective ($\operatorname{\mathsf{Spec}} k$ is a point !), it is an étale covering. We therefore have shown that every non-empty smooth scheme over a field k has a section *locally for the étale topology on* $\operatorname{\mathsf{Spec}} k$.

This fact admits the following generalization, whose proof is a little bit more technical, but essentially no more difficult, than the preceding one: *if $\varphi : Y \to X$ is a smooth,* surjective *morphism between two schemes, then it does admit a section locally for the étale topology on X.*

Some comments.

1) The above claim is definitely false if one restricts to Zariski topology, which is too coarse. Let us give two counter-examples, the first one being 'arithmetic' (*i.e.* related to the fact that the ground field is not algebraically closed) and the second one being 'geometric' (*i.e.* over an algebraically closed field).

The arithmetic example. Let X be the non-trivial real conic, that is, the projective $\mathbb{R}$-scheme defined in homogeneous coordinates by the equation

$$T_0^2 + T_1^2 + T_2^2 = 0;$$

it can be easily shown that X is smooth, but $X \to \mathsf{Spec}\,\mathbb{R}$ doesn't admit any section, because $X(\mathbb{R}) = \emptyset$ (as $\mathsf{Spec}\,\mathbb{R}$ is a point, to have a section locally for the Zariski topology on $\mathsf{Spec}\,\mathbb{R}$ just means to have a section). Of course, $X(\mathbb{C}) \neq \emptyset$.

The geometric example. Set $Y = \mathsf{Spec}\,\mathbb{C}[U, V, S, T]/(SU^2 + TV^2 - 1)$, let $\varphi : Y \to \mathbb{A}^2_{\mathbb{C}}$ be the morphism defined by S and T, and let X be the complement of the origin in $\mathbb{A}^2_{\mathbb{C}}$. One can check that φ is smooth over X ; but it doesn't admit sections locally on X.

Indeed, one can even prove that if X' is any non-empty open subset of X, then φ doesn't admit a section over X': if it did, the restriction of this section to the generic fiber Y_η of φ would define a $\mathbb{C}(S, T)$-point on Y_η, that is, a solution of the equation $SU^2 + TV^2 = 1$ in $\mathbb{C}(S, T)$ (variables are U and V); but (exercise) there are no such solutions.

Let us now show concretely that φ has locally a section on X for the étale topology. For that purpose, let us denote by ψ_1 the map

$$\mathbb{A}^2_{\mathbb{C}} = \mathsf{Spec}\,\mathbb{C}[S', T] \to \mathbb{A}^2_{\mathbb{C}} = \mathsf{Spec}\,\mathbb{C}[S, T]$$

that is induced by $S \mapsto (S')^2$ and $T \to T$ (at the level of points, it is simply $(s', t) \mapsto ((s')^2, t)$), and by ψ_2 the map

$$\mathbb{A}^2_{\mathbb{C}} = \mathsf{Spec}\,\mathbb{C}[S, T'] \to \mathbb{A}^2_{\mathbb{C}} = \mathsf{Spec}\,\mathbb{C}[S, T]$$

that is induced by $S \mapsto S$ and $T \to (T')^2$ (at the level of points, it is simply $(s, t') \mapsto (s, (t')^2)$).

Let X_1 (resp. X_2) be the open affine subscheme of $\mathbb{A}^2_{\mathbb{C}}$ defined as the invertibility locus of S (resp. T). It is easily seen that $\psi_i^{-1}(X_i) \to X_i$ is a finite étale cover of degree 2 for $i = 1, 2$; as $X = X_1 \cup X_2$, the family $(\psi_1^{-1}(X_1) \to X, \psi_2^{-1}(X_2) \to X)$ is an étale covering of X.

Now for every i the map $Y \times_{\mathbb{A}^2_{\mathbb{C}}} \psi^{-1}(X_i) \to \psi^{-1}(X_i)$ admits a section:

• if $i = 1$ one can consider the map defined as the level of rings by the formulas $U \mapsto (1/S'), V \mapsto 0$ (at the level of points, this map corresponds to $(s', t) \mapsto (1/s', 0, s', t)$);

• if $i = 2$ one can consider the map defined as the level of rings by the formulas $U \mapsto 0, V \mapsto 1/T'$ (at the level of points, this map corresponds to $(s, t') \mapsto (0, 1/t', s, t')$).

2) In the complex analytic case, it is obvious that a smooth surjective map admits sections locally on the target (because such a map is locally on the source and on the target isomorphic to a projection $X \times \mathbb{D}^n \to X$ for some $n \geqslant 0$, where $\mathbb{D}$ is the open unit disc); hence we see that the étale topology is, for some purposes, a good algebraic analog of the classical topology over $\mathbb{C}$ – at least far better than the Zariski topology.

The fundamental group of a scheme

As we will see, Grothendieck's theory of the fundamental group provides, through the notion of a finite étale map, a very natural frame containing both Galois theory and the theory of (finite) *topological* coverings of complex algebraic varieties, which may help to get a geometric intuition of some *a priori* purely algebraic field-theoretic notions.

In consistency with the fact that the étale avatar of a one-point space is supposed to be the spectrum of a separably closed field, one defines a *geometric point* on a scheme X as a morphism $\mathsf{Spec}\ F \to X$, for F a separably closed field; equivalently, such a point can be defined as a datum of a (true) point x of the scheme X, together with an embedding $\kappa(x) \hookrightarrow F$ for some separably closed field F.

Let X and Y be two connected, non-empty schemes and let $Y \to X$ be finite and étale. It can be shown that $G := \mathsf{Aut}\ Y/X$ is a finite group.

One says that $Y \to X$ is *Galois* if X is *scheme-theoretically* equal to the quotient Y/G; it means exactly that every arrow $Y \to Z$ which is G-invariant goes in a unique way through an arrow $X \to Z$, and if $Y = \mathsf{Spec}\ A$, this is equivalent to the fact that $X = \mathsf{Spec}\ A^G$.

As an example, if k is a field and if L is a finite separable extension, then the finite étale map $\mathsf{Spec}\ L \to \mathsf{Spec}\ k$ is Galois if and only if L is a Galois extension of k in the usual sense.

Now, let us choose a *geometric point* $\overline{x}$ on X (one should think to it as a 'basepoint'). One can associate to it a profinite group $\pi_1(X, \overline{x})$ which is called the *fundamental group* of $(X, \overline{x})$.

The group $\pi_1(X, \overline{x})$ encodes the theory of finite étale covers of X. This quite vague claim can be given a precise meaning, involving an equivalence of categories, which we won't explain in full detail here; let us nevertheless give some precisions.

- To every open subgroup U of $\pi_1(X, \overline{x})$ is associated a connected, non-empty finite étale cover $\mathsf{Cov}\ U$ of X.
- If U and V are two open subgroups of $\pi_1(X, \overline{x})$ with $U \subset V$, the arrow $\mathsf{Cov}\ U \to X$ goes through a natural finite étale cover $\mathsf{Cov}\ U \to \mathsf{Cov}\ V$.
- If U is an open subgroup U of $\pi_1(X, \overline{x})$ then $\mathsf{Cov}\ U$ is Galois if and only if $U \lhd \pi_1(X, \overline{x})$, and if it is the case there is a natural isomorphism $\mathsf{Aut}\ (\mathsf{Cov}\ U/X) \simeq \pi_1(X, \overline{x})/U$.
- Every connected, non-empty, finite étale cover is isomorphic to $\mathsf{Cov}\ U$ for some U.
- $\pi_1(X, \overline{x})$ is naturally isomorphic to

$$\varprojlim_{U \lhd \pi_1(X, \overline{x})} \mathsf{Aut}\ (\mathsf{Cov}\ U/X).$$

Let us mention that if $\overline{y}$ is another geometric point of X, then $\pi_1(X, \overline{y})$ is *non-canonically* isomorphic to $\pi_1(X, \overline{x})$. Therefore it may happen that one speaks of *the* fundamental group of X, without mentioning any geometric point (and that one notes it $\pi_1(X)$)– but this is a little bit abusive and sometimes

dangerous, exactly like speaking of the fundamental group of a connected, non-empty topological space, without having pointed it.

Example. Suppose $X = \mathsf{Spec}\, k$, with k a field, and $\overline{x} = \mathsf{Spec}\, k^s$ for some separable closure k^s of x. Then $\pi_1(X, \overline{x})$ is nothing but $\mathsf{Gal}\,(k^s/k)$, and what is written above is (part of) Galois correspondence.

Remark. We see that choosing a separable closure of a given field is, in some sense, analog to choosing a basepoint on a (connected, non-empty) topological space.

Functoriality. Let $Y \to X$ be a morphism between two connected, non-empty schemes, and let $\overline{y}$ be a geometric point of Y. It defines by composition with $Y \to X$ a geometric point $\overline{x}$ of X; the map $Y \to X$ then induces in a natural way a continuous morphism $\pi_1(Y, \overline{y}) \to \pi_1(X, \overline{x})$.

Various comparison theorems. In what follows, we will implicitly use the following fact: let k be an algebraically closed field and let L be an algebraically closed extension of k. If X is a connected, non-empty algebraic variety over k, then the L-algebraic variety $X_L := X \times_k L$ is non-empty and connected too; if $k = \mathbb{C}$, then $X(\mathbb{C})$ is also non-empty and connected for the transcendent topology.

Our aim is now to describe the behaviour of the fundamental group when one increases the *algebraically closed* ground field, and how the Grothendieck fundamental group of an algebraic variety over $\mathbb{C}$ and its classical (topological) fundamental group are related.

Theorem 1. *Let k be an algebraically closed field and let X be a connected, non-empty algebraic variety over k. Let L be an algebraically closed extension of k, let $\overline{\xi}$ be a geometric point of X_L and let $\overline{x}$ be the geometric point on X defined by composition of $\overline{\xi}$ with $X_L \to X$.*

Assume that k is of char. 0 or that the variety X is proper (e.g. projective). The natural map $\pi_1(X_L, \overline{\xi}) \to \pi_1(X, \overline{x})$ is then an isomorphism.

Some comments.

1) In a more concrete way, what theorem 1 says is that if char $k = 0$ or if X is proper then *every finite étale covering of X_L can already be defined over k.*

2) Theorem 1 is definitely false if char. $k = p > 0$ and if X is not proper, in which case the fundamental group will become huger and huger by enlarging the algebraically closed ground field; or, in other words, there will be plenty of étale covers of X_L not definable over k as soon as $L \neq k$.

Let us give an example of such a cover: if $\lambda \in L - k$, the étale cover

$$\mathsf{Spec}\, L[X, T]/(T^p - T - \lambda X) \to \mathsf{Spec}\, L[X] = \mathbb{A}^1_L$$

can not be defined over k.

3) Let us assume that char. $k = p > 0$; for every profinite group G, we denote by $G_{\neq p}$ the largest profinite quotient of G all of whose finite quotients are of prime-to-p-order. One can then prove, with the notations of th. 1 and without any properness assumption, that $\pi_1(X_L, \overline{\xi})_{\neq p} \to \pi_1(X, \overline{x})_{\neq p}$ is an isomorphism. This means that every *prime-to-p* Galois cover of X_L can be defined over k.

Let us come now to the comparison between the étale and topological fundamental groups.

Theorem 2. *Let X be a connected, non-empty complex algebraic variety, let $x \in X(\mathbb{C})$ and let us denote by x the corresponding geometric point $\mathsf{Spec}\,\mathbb{C} \to X$ of the scheme X. There is a natural isomorphism between $\pi_1(X, x)$ and the profinite completion of $\pi_{1,\mathrm{top}}(X(\mathbb{C}), x)$.*

Some comments. In a more concrete way, what theorem 2 says is that *every finite topological cover of $X(\mathbb{C})$ is algebraizable*; this is what is classically called 'Riemann's existence theorem'.

A consequence of th. 1 and 2. Let X be a connected, non-empty algebraic variety over $\mathbb{C}$ and let σ be any (possibly non-continuous) automorphism of $\mathbb{C}$. The corresponding twisted variety X_σ coincides with X as a scheme; only its structure map to $\mathsf{Spec}\,\mathbb{C}$ has been changed (through σ).

By applying th. 2 to X and X_σ, and by using the fact that they coincide as schemes, one sees that *the groups $\pi_{1,\mathrm{top}}(X(\mathbb{C}))$ and $\pi_{1,\mathrm{top}}(X_\sigma(\mathbb{C}))$ have isomorphic profinite completions.*

Note that this assertion about profinite completions simply means that $X(\mathbb{C})$ and $X_\sigma(\mathbb{C})$ 'have the same finite topological covers'.

Remak. Serre has given an example ([7]) of a complex algebraic variety X and an automorphism σ of $\mathbb{C}$ such that $\pi_1(X(\mathbb{C}))$ and $\pi_1(X_\sigma(\mathbb{C}))$ are *not* isomorphic though they have, by the above, the same profinite completions. Roughly speaking, $X(\mathbb{C})$ and $X_\sigma(\mathbb{C})$ have the same finite topological covers, but not the same general topological covers.

The fundamental group of a projective curve. Let k be an algebraically closed field and let X be a smooth, irreducible, projective k-curve.

a) If $X = \mathbb{P}_k^1$ then $\pi_1(X) = 0$.

b) If X is of positive genus g and if char. $k = 0$, then $\pi_1(X)$ admits, *as a profinite group*, a presentation

$$< a_1, b_1, \ldots, a_g, b_g > /[a_1, b_1][a_2, b_2] \ldots [a_g, b_g].$$

c) If X is of positive genus g and if char. $k = p > 0$, then $\pi_1(X)_{\neq p}$ admits, *as a profinite group with only prime-to-p finite quotients*, a presentation

$$< a_1, b_1, \ldots, a_g, b_g > /[a_1, b_1][a_2, b_2] \ldots [a_g, b_g].$$

About the proofs of the above results. Claim a) follows easily from Hurwitz's formula; claim b) is proven by reducing, through th. 1, first of all to the case where k is contained in $\mathbb{C}$, and then to the case where $k = \mathbb{C}$, in which one just has to apply th. 2 (and the transcendental standard description of the fundamental group of a Riemann surface).

Claim c) is far more subtle and was the first success of Grothendieck's approach to algebraic geometry.

First step. One exhibits a complete, discrete valuation ring A with residue field k and with quotient field K of char. 0: this is the 'Witt vectors construction'.

Second step. One extends X to a flat, smooth, projective A-scheme $\mathscr{X}$, as follows: first, one makes a formal extension, which is possible because the obstruction to do it lives in a Zariski H^2 group of X, which is 0 by a dimension argument; and then, one algebraizes this formal extension, which is again possible thanks to dimension 1.

The core of the proof. Let $\overline{K}$ be an algebraic closure of K. A comparison theorem which is due to Grothendieck says that

$$\pi_1(\mathscr{X} \times_A \overline{K})_{\neq p} \simeq \pi_1(X)_{\neq p} \; ;$$

it is the scheme-theoretic analog of the following classical fact: if $\varphi : Y \to X$ is a smooth, surjective and proper map with connected fibers between complex analytic spaces, then $x \mapsto \pi_{1\mathrm{top}}(\varphi^{-1}(x))$ is locally constant on X.

Conclusion. The required conclusion now follows by applying assertion b) to the curve $\mathscr{X} \times_A \overline{K}$.

5 Étale morphisms of analytic spaces

General reference: [1].

We fix a complete, non-Archimedean field k.

In order to simplify exposition, we have chosen to give the definition of a finite étale map before that of a general étale map. But fortunately, one can check that étale maps which are moreover finite are nothing but the 'finite étale' maps defined in a first time: though it is not introduced in the logical way, our terminology is consistent.

Definition. Let $\varphi : Y \to X$ be a morphism between two k-analytic spaces. We say that φ is *finite étale* if for every affinoid domain $U = \mathscr{M}(\mathscr{A})$ of X, the pre-image V of U on Y is an affinoid domain of Y equal to $\mathscr{M}(\mathscr{B})$ for some finite étale $\mathscr{A}$-algebra $\mathscr{B}$.

Remark. It is sufficient to check the above property on an affinoid G-covering of X.

Let us now come to the definition of general étale maps.

Definition. Let $\varphi : Y \to X$ be a morphism between two k-analytic spaces, let $y \in Y$ and let $x = \varphi(y)$. One says that φ is *étale at y* if there exists an open neighborhood U of x and an open neighborhood V of y such that φ induces a finite étale map (in the above sense) from V to U.

We say that a morphism $Y \to X$ between two k-analytic spaces is *étale* if it is étale at every point of Y.

Remark. Be aware that one can not, in general, take V being equal to $\varphi^{-1}(U)$.

Zariski topology is too coarse to make the above definition work in scheme theory. Indeed, let us assume that $p \neq 2$; let Y be the invertibility locus of $T - 1$ on $\mathbb{G}_m$ and let φ be the map $Y \to \mathbb{G}_m$ induced by the finite étale arrow $\mathbb{G}_m \overset{z \mapsto z^2}{\longrightarrow} \mathbb{G}_m$. Let y be the closed point of Y at which $T + 1 = 0$; its image x is the closed point of $\mathbb{G}_m$ at which $T = 1$, and one has $\kappa(y) = \kappa(x) = k$.

Then though φ is étale at y, it is not possible to find an open neighborhood V of y in Y and an open neighborhood U of x in $\mathbb{G}_m$ such that φ induces a finite étale map $V \to U$. Indeed, if it were possible, then by checking what happens at the generic point of the target, one sees that the degree of $V \to U$ would be equal to 2, contradicting the fact that $\varphi^{-1}(x) = \{y\}$ with $\kappa(y) = \kappa(x)$, hence is of degree *one* over $\kappa(x)$.

First properties. The étale locus of a given map is an open subset of its source; if $\varphi : Y \to X$ is a morphism of k-analytic spaces, then φ is étale at a point $y \in Y$ if and only if φ is finite, flat and unramified at y.

We are now going to give some basic properties of étale maps; for the sake of simplicity, we restrict ourselves to *global* étale morphisms; but most of those results also have a local counterpart.

Every étale map is open and *boundaryless*. If $Z \to Y$ and $Y \to X$ are étale, then the composite map $Z \to X$ is étale too. If $Y \to X$ is étale and if $X' \to X$ is *any* morphism, then $Y \times_X X' \to X'$ is étale. If $Y \to X$ and $Z \to X$ are étale, then *any* morphism $Y \to Z$ making the diagram

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ & \searrow & \downarrow \\ & & X \end{array}$$

commute is étale.

Examples.

- Every open immersion is étale. Be careful: the embedding of a non-open analytic domain (*e.g.* the closed unit ball in an affine space) is *not* étale because it has non-empty boundary.

- If $Y \to X$ is a morphism between algebraic k-varieties, then $Y^{an} \to X^{an}$ is étale if and only if $Y \to X$ is étale.

- A very important class of étale maps is that of *étale covers*. A morphism $\varphi : Y \to X$ is said to be an étale cover if X admits an open covering (X_i) such that for every i one can write

$$\varphi^{-1}(X_i) = \coprod_{j \in J_i} Y_{i,j}$$

with each $Y_{i,j}$ finite étale over X_i (and with the index set J possibly infinite). Let us now give three examples of étale covers.

  1) *An example of finite étale cover.* Let r and R be two positive real numbers with $0 < r < R$, let n be a positive integer which is non-zero in k, and let Y (resp. X) be the open subset of $\mathbb{A}_k^{1,an}$ defined by the conditions $r < |T| < R$ (resp. $r^n < |T| < R^n$). The map

  $$Y \to X, z \mapsto z^n$$

  is a finite étale cover of degree n.

  2) *A topological cover.* Every topological cover of an analytic space inherits a structure of an analytic space – and becomes an étale cover of the target. Let us give now a fundamental example.

  Assume that k is algebraically closed and non-trivially valued, and let E be a k-elliptic curve with *bad* reduction. By a celebrated theorem of Tate (which was written in the rigid-analytic language and was actually the first motivation for developing such a theory), there exist $q \in k^*$ with $|q| < 1$ and an isomorphism $E^{an} \simeq \mathbb{G}_m^{an}/q^{\mathbb{Z}}$.

  The *uniformization* arrow $\mathbb{G}_m^{an} \to E^{an}$ is a topological cover – this is even a universal cover of E^{an} –, and in particular it is an étale cover.

3) *The logarithm.* Let $\mathbb{C}_p$ be the completion of an algebraic closure of $\mathbb{Q}_p$, let X be the open unit $\mathbb{C}_p$-disc with center 1, and let log be the usual logarithm defined on X by the formula

$$\log(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots$$

Then log induces an infinite étale cover $X \to \mathbb{A}^{1,an}_{\mathbb{C}_p}$; it is not a topological cover (indeed, $\mathbb{A}^{1,an}_{\mathbb{C}_p}$ is topologically simply connected).

- *Some comments on the above examples.* The two last examples show that, countrary to what happens in algebraic geometry, one can exhibit in analytic geometry natural *infinite* étale covers between connected (and very basic) spaces.

- *The notion of a Galois cover.* As in the algebraic setting, an étale cover between analytic spaces is said to be Galois if the target is the quotient of the source (in a suitable meaning, analog to the one we used in scheme-theory) by the automorphism group of the cover.

  Example 1) above is Galois as soon as k is algebraically closed, and its Galois group is then equal to $\mu_n(k)$ – the latter beeing itself non-canonically isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

  Example 2) above is Galois, with Galois group $q^{\mathbb{Z}}$, hence isomorphic to $\mathbb{Z}$.

  Example 3) above is Galois, with Galois group $\mu_{p^\infty}(\mathbb{C}_p) := \bigcup_n \mu_{p^n}(\mathbb{C}_p)$.

The fibers of an étale map. Let $\varphi : Y \to X$ be an étale morphism between two k-analytic spaces and let $x \in X$. Its fiber $\varphi^{-1}(x)$ is discrete; this is a disjoint (possibly infinite) union $\coprod \mathscr{M}(k_i)$ where every k_i is a finite separable extension of $\mathscr{H}(x)$.

Moreover, if y is a preimage of x, the germ of morphism $(Y, y) \to (X, x)$ *only depends on the finite separable field extension $\mathscr{H}(x) \hookrightarrow \mathscr{H}(y)$.*

Note a very useful particular case of the latter assertion: *the map φ is a local isomorphism at y if and only if $\mathscr{H}(x) = \mathscr{H}(y)$.*

If k is algebraically closed and if x is a k-point, then $\mathscr{H}(y) = \mathscr{H}(x)$ for every preimage y of x, hence φ is a local isomorphism at every pre-image of x.

But this is false in general, that is, if k is not algebraically closed or if x is not a k-point; let us give a simple, explicit example. For every $s > 0$ denote by η_s the point of $\mathbb{A}^{1,an}_k$ defined by the semi-norm $\sum a_i T^i \mapsto \max |a_i| s^i$. Then in example 1) above for every $s \in]r; R[$ the pre-image of η_{s^n} is $\{\eta_s\}$, and the field extension $\mathscr{H}(\eta_{s^n}) \hookrightarrow \mathscr{H}(\eta_s)$ induced by our étale map is of degree n, hence is non-trivial as soon as $n > 0$.

The étale topology on an analytic space.

Let X be an analytic space. We have defined what an étale morphism of analytic spaces is. This allows us to define the étale site $X_{\text{ét}}$ and to speak about the étale topology on X (see just before section 4); we now want to give an example of a property which is satisfied locally for the étale topology on X, but not in general for its usual topology.

Let $Y \to X$ be a morphism of analytic spaces, and let $y \in Y$. We will say that $Y \to X$ is *smooth at* y if there exists an integer n, an open neighborhood U of y and a factorization of $U \to X$ through an étale map $U \to \mathbb{A}_X^n$; note that if $Y \to X$ is étale at y, it is smooth at y (take $n = 0$ and U beeing the étale locus of $Y \to X$).

As in scheme theory, the following is true: *if $\varphi : Y \to X$ is a smooth, surjective morphism between two analytic spaces, then it does admit a section locally for the étale topology on X.*

Comment. The above claim is definitely false if one restricts to the usual topology on X, which is too coarse.

Fundamental groups of an analytic space. Let X be a non-empty, connected k-analytic space and let $\overline{x}$ be a geometric point of x, that is, a true point x of X together with an embedding of $\mathscr{H}(x)$ into a separably closed field. To the datum $(X, \overline{x})$ one can associate in a functorial way various topological groups, which are in general neither discrete nor profinite, each of which encoding part of the theory of étale covers of X.

As an example, the theory of the finite étale covers (resp. topological étale covers, resp. all étale covers) of X is encoded in a profinite (resp. discrete, resp. topological) group which is denoted by $\pi_{1,\mathrm{f}}(X, \overline{x})$ (resp. $\pi_{1,\mathrm{top}}(X, \overline{x})$, resp. $\pi_1(X, \overline{x})$); Yves André has introduced another one, which is denoted by $\pi_{1,\mathrm{temp}}(X, \overline{x})$ and which classifies the so-called *tempered* étale covers, that is, those which become topological after a finite étale base change; this group has turned out to be of deep arithmetical interest.

First remark. Both profinite completions of $\pi_1(X, \overline{x})$ and $\pi_{1,\mathrm{temp}}(X, \overline{x})$ are naturally isomorphic to $\pi_{1,\mathrm{f}}(X, \overline{x})$.

Second remark. Each of those groups only depends, *up to a non-canonical isomorphism*, on X and not on the geometric point $\overline{x}$; therefore it may happen that one doesn't mention the latter.

Examples.

• If X is an algebraic variety over k and if $\overline{x}$ is a geometric point of X^{an}, then it induces a geometric point $\overline{\xi}$ of X, and there is a natural continuous homomorphism $\pi_{1,\mathrm{f}}(X^{an}, \overline{x}) \to \pi_1(X, \overline{\xi})$.

If char. $k = 0$ then $\pi_{1,\mathrm{f}}(X^{an}, \overline{x}) \to \pi_1(X, \overline{\xi})$ is an isomorphism; if char. $k = p > 0$ then it only induces an isomorphism $\pi_{1,\mathrm{f}}(X^{an}, \overline{x})_{\neq p} \to \pi_1(X, \overline{\xi})_{\neq p}$. Roughly speaking, if char. $k = 0$ (resp. char. $k = p > 0$) then every finite (resp. finite and prime-to-p) Galois cover of X^{an} is algebraic.

• Suppose that k is algebraically closed and that X is an open disc over k; let p be the characteristic of the residue field $\widetilde{k}$. If $p = 0$ (resp. if $p > 0$) then every finite (resp. finite and prime-to-p) Galois cover of X is trivial; hence $\pi_{1,\mathrm{f}}(X) = 0$ (resp. $\pi_{1,\mathrm{f}}(X)_{\neq p} = 0$).

• Suppose that k is algebraically closed and that X is an open annulus over k; let p be the characteristic of the residue field $\widetilde{k}$. If $p = 0$ (resp. if $p > 0$) then every finite (resp. finite and prime-to-p) Galois cover of X is Kummer, that is, as in example 1) above; hence $\pi_{1,\mathrm{f}}(X) \simeq \widehat{\mathbb{Z}}$ (resp. $\pi_{1,\mathrm{f}}(X)_{\neq p} \simeq \widehat{\mathbb{Z}}_{\neq p} = \prod_{\ell \neq p} \mathbb{Z}_\ell$).

- Suppose that k is algebraically closed and let E be an elliptic curve over k. If E has good reduction then

$$\pi_{1,\mathrm{f}}(E^{an}) \simeq \widehat{\mathbb{Z}}^2 \ ; \pi_{1,\mathrm{top}}(E^{an}) = 0 \quad \text{and} \quad \pi_{1,\mathrm{temp}}(E^{an}) \simeq \widehat{\mathbb{Z}}^2 \ ;$$

if E has bad reduction then

$$\pi_{1,\mathrm{f}}(E^{an}) \simeq \widehat{\mathbb{Z}}^2 \ ; \pi_{1,\mathrm{top}}(E^{an}) \simeq \mathbb{Z} \quad \text{and} \quad \pi_{1,\mathrm{temp}}(E^{an}) \simeq \widehat{\mathbb{Z}} \times \mathbb{Z} \ .$$

- The topological group $\pi_1(\mathbb{P}^{1,an}_{\mathbb{C}_p})$ is huge.

6 Étale sheaves and étale cohomology on schemes

General reference: [11], [12], [13], [14] (for the ℓ-adic formalism) and [6].

Let us begin with a remark: as open immersions are étale, any open covering of a scheme X is an étale covering of X; therefore every sheaf on $X_{\mathrm{ét}}$ is, when restricted to the category of open subsets of X, a usual sheaf.

Examples of sheaves on $X_{\mathrm{ét}}$

The constant sheaves. Let A be a set. The constant sheaf $\underline{A}$ associated with A admits the following concrete description: for every $U \in X_{\mathrm{ét}}$, the set $\underline{A}(U)$ is the set of locally constant functions from U to A; if U is locally connected (*e.g* U is an algebraic variety over a field) then it can also be described as $A^{\pi_0(U)}$. Il A is a an abelian group, then $\underline{A}$ is a sheaf of abelian groups. As usual, we will most of the time write A instead of $\underline{A}$, and call it the *constant sheaf A*.

The sheaf of global functions. The assignment $U \mapsto \mathscr{O}_U(U)$ is a sheaf on $X_{\mathrm{ét}}$ (we already knew that it was a sheaf for the Zariski topology). This means that for every étale $U \in X_{\mathrm{ét}}$, and for every étale covering $(U_i \to U)$, any family (s_i) with $s_i \in \mathscr{O}_{U_i}(U_i)$ for all i and $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$ for all (i,j) arises from a uniquely determined $s \in \mathscr{O}_U(U)$.

Let us see what it does mean in an interesting particular case. Suppose that $X = U = \mathsf{Spec}\, k$ for some field k and that $(U_i \to U)$ is a *single* map $\mathsf{Spec}\, L \to \mathsf{Spec}\, k$ for some finite Galois extension L of k with Galois group G; let p_1 and p_2 be the two projections

$$\mathsf{Spec}\, L \times_{\mathsf{Spec}\, k} \mathsf{Spec}\, L \to \mathsf{Spec}\, L.$$

The claim says that every $l \in L$ such that $p_1^*(l) = p_2^*(l)$ belongs to k.

Now one has an isomorphism between $L \otimes_k L$ and the ring of G-indexed finite sequences of elements of L, which sends $(l \otimes \lambda)$ to $(lg(\lambda))_{g \in G}$; modulo this isomorphism, $p_1^*(l) = (l)_g$ and $p_2^*(l) = (g(l))_g$ for every $l \in L$.

Therefore one has $p_1^*(l) = p_2^*(l)$ if and only if $g(l) = l$ for every $g \in G$. Hence, our claim simply rephrases one of the fundamental results of Galois theory, that is, the equality $k = L^G$.

Some comments. We thus see that the theory of étale sheaves helps thinking to the equality $k = L^G$ in a quite geometrical way. More precisely, it follows

from above that the fact, for an element of L, to be G-invariant, can be thought as a 'coincidence' condition; and the fact that such an element comes from k can be thought as the result of a glueing process this coincidence condition has made possible.

This geometric vision of Galois phenomena has turned out to be very fruitful, especially as far as the so-called *descent theory* is concerned. The typical questions addressed by this theory is: *does a given object defined on L and equipped with a Galois action come from k ?* It is very helpful to think to it as a glueing problem (of objects, no more of functions), the existence of the Galois action corresponding precisely to the datum of 'a compatible system of isomorphisms on the intersections'.

Coherent sheaves. More generally, let $\mathscr{F}$ be a coherent (Zariski) sheaf on X. Remind that by the very definition of $X_{\text{ét}}$ every object U of $X_{\text{ét}}$ comes equipped with an étale morphism $\pi : U \to X$; let us assign to such an U the $\mathscr{O}_U(U)$-module $\pi^*\mathscr{F}(U)$. One can prove that this is a sheaf on $X_{\text{ét}}$, which we denote by $\mathscr{F}_{\text{ét}}$; note that the sheaf we have considered above is then nothing but $(\mathscr{O}_X)_{\text{ét}}$.

One can prove that for every coherent sheaf $\mathscr{F}$ on X and every integer i, the groups $\mathrm{H}^i(X_{\text{Zar}}, \mathscr{F})$ and $\mathrm{H}^i(X_{\text{ét}}, \mathscr{F}_{\text{ét}})$ are canonically isomorphic. Hence as far as cohomology of coherent sheaves is concerned, étale cohomology doesn't contain more information than Zariski cohomology.

Remark. One often writes $\mathscr{F}$ instead of $\mathscr{F}_{\text{ét}}$, when the context is sufficiently clear to indicate that one doesn't work with the Zariski sheaf $\mathscr{F}$ but with the étale sheaf naturally associated with it. We will sometimes make such an abuse – and write $\mathscr{O}_X$ for the sheaf $U \mapsto \mathscr{O}_U(U)$ on $X_{\text{ét}}$, as an example; the latter is also usually sometimes denoted by $\mathbb{G}_a$.

The multiplicative group. The assignment $U \mapsto \mathscr{O}_U(U)^*$ is a sheaf of abelian groups on $X_{\text{ét}}$; it is a subsheaf (of multiplicative monoids) of $\mathscr{O}_X$ which is often denoted by $\mathbb{G}_m$.

The 90'th theorem by Hilbert says that there is a natural isomorphism $\mathrm{H}^1(X_{\text{Zar}}, \mathbb{G}_m) \simeq \mathrm{H}^1(X_{\text{ét}}, \mathbb{G}_m)$; this means that any 'étale line bundle' - that is, an étale sheaf of $\mathscr{O}_X$-modules locally isomorphic to $\mathscr{O}_X$ for the étale topology on X– is already locally isomorphic to $\mathscr{O}_X$ for the *Zariski* topology.

More generally, for every n an étale sheaf of $\mathscr{O}_X$-modules which is locally isomorphic to $\mathscr{O}_X^n$ for the étale topology on X is already locally isomorphic to $\mathscr{O}_X^n$ for the *Zariski* topology[4]. When $X = \mathsf{Spec}\, k$ for k a field, the latter claim has a very concrete meaning: if L is a finite Galois extension of k and if G is the Galois group of L, then every finite dimensional L-vector space endowed with an action of G is G-isomorphic to $L \otimes_k V$ for some k-vector space V (in other words, it admits a basis over L whose vectors are invariant under G). This is an example of the aforementioned descent theorems.

The sheaf of roots of unity. Let n be an integer. Let μ_n be the presheaf on $X_{\text{ét}}$ defined by the assignment $U \mapsto \{f \in \mathbb{G}_m(U), f^n = 1\}$; it easily seen to be a subsheaf of $\mathbb{G}_m$; we denote it by μ_n (the 'sheaf of n-th root of unity').

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[4]This can also be expressed as a comparison between étale and Zariski H^1, but with coefficients GL_n – which is not an *abelian* group; we won't speak more about it here.

Remark. If the ambient ground scheme need to be precised, one will write $A_X, \mathbb{G}_{m,X}, \mathsf{GL}_{n,X}$ and $\mu_{n,X}$ instead of $A, \mathbb{G}_m, \mathsf{GL}_n$ and μ_n.

An important example of locally constant sheaf. Let us assume that the integer n is invertible on X and let $\Phi \in \mathbb{Z}[X]$ be the n-th cyclotomic polynomial. One checks that for every open affine subscheme $U = \mathsf{Spec}\ A$ of X the finite map $V := \mathsf{Spec}\ A[T]/\Phi \to U$ is an étale cover, and that the map $\overline{m} \mapsto \overline{T}^m$ induce an isomorphism of sheaves $(\mathbb{Z}/n\mathbb{Z})_V \simeq \mu_{n,V}$; hence the restriction of μ_n to V is constant, whence we deduce that μ_n *is locally constant for the étale topology on X.*

Locally constant sheaves and representations of π_1. Let A be a finite set (resp. a finite abelian group) and assume that X is a connected and non-empty scheme; let $\overline{x}$ be a geometric point of X. There is exactly the same phenomenon as in usual topology: one can establish a natural dictionary between sheaves (resp. sheaves of abelian groups) on $X_{\text{ét}}$ which are locally isomorphic to the constant sheaf A and discrete (*i.e.* with open stabilizers) actions of $\pi_1(X, \overline{x})$ on A, up to conjugation by a permutation (resp. an automorphism) of A.

Let us give a very simple example of this in the arithmetical context. Let $X = \mathsf{Spec}\ k$ with k a field, and let $\overline{x}$ be given by a separable closure k^s of k whose Galois group is denoted by G; fix an integer n which is not zero in k, and let ζ be a primitive n-th root of 1 in k^s. The datum of ζ induces an isomorphism between the groups $\mathbb{Z}/n\mathbb{Z}$ and $\mu_n(k^s)$, whence a discrete action of G on $\mathbb{Z}/n\mathbb{Z}$; this is precisely the action that corresponds to the sheaf μ_n on $\mathsf{Spec}\ k_{\text{ét}}$, which has been seen to be locally isomorphic to the constant sheaf $\mathbb{Z}/n\mathbb{Z}$.

Cohomology of a locally constant sheaf. Let us return to a general, non-empty connected scheme X endowed with a geometric point $\overline{x}$; let A be a finite abelian group and let $\mathscr{F}$ be an étale sheaf on X which is locally isomorphic to A; it corresponds to a discrete action of $\pi_1(X, \overline{x})$ on A.

It can then be proven that $\mathrm{H}^1(X_{\text{ét}}, \mathscr{F}) \simeq \mathrm{H}^1(\pi_1(X, \overline{x}), A)$, where A is endowed with the aforementioned $\pi_1(X, \overline{x})$-action (see section 3 for the definition of $\mathrm{H}^1(\pi_1(X, \overline{x}), .)$).

The particular case where $\mathscr{F} = A$. In that case, the action of $\pi_1(X, \overline{x})$ on A is trivial, and $\mathrm{H}^1(\pi_1(X, \overline{x}), A)$ is nothing but $\mathsf{Hom}_{\text{cont}}(\pi_1(X, \overline{x}), A)$ (the topology on A being the discrete one).

The group $\mathsf{Hom}_{\text{cont}}(\pi_1(X, \overline{x}), A)$ classifies up to isomorphism the couples $(Y \to X, \iota)$ where $Y \to X$ is a connected Galois cover and ι an embedding $\mathsf{Aut}\ (Y/X) \hookrightarrow A$; that is therefore also the case for $\mathrm{H}^1(X_{\text{ét}}, A)$.

A notation. If X is a scheme and if ℓ is a prime number, we will denote by $\mathsf{Sh}(X, \ell)$ the category of sheaves on $X_{\text{ét}}$ which are locally isomorphic to a constant sheaf associated with a finite abelian ℓ-group.

Pull-backs. Let $f : Y \to X$ is a morphism of schemes. To every sheaf $\mathscr{F}$ on $X_{\text{ét}}$ can be associated in a natural way a sheaf $f^*\mathscr{F}$ on $Y_{\text{ét}}$; If $\mathscr{F} = A_X$ for some set A then $f^*\mathscr{F} = A_Y$; one has $f^*\mu_{n,X} = \mu_{n,Y}$ for every n invertible on X.

If $\mathscr{F}$ is locally isomorphic to a constant sheaf A_X, then $f^*\mathscr{F}$ is locally isomorphic to A_Y (note in particular that if $\mathscr{F} \in \mathsf{Sh}(X, \ell)$ for some prime number ℓ, then $f^*\mathscr{F} \in \mathsf{Sh}(Y, \ell)$). If one assumes moreover that A is finite and that Y and X are non-empty and connected, one has a nice description of $f^*\mathscr{F}$,

as follows. Let $\overline{y}$ be a geometric point of Y and let $\overline{x}$ denote the corresponding geometric point of X. The locally constant sheaf $\mathscr{F}$ is given by an action of $\pi_1(X, \overline{x})$ on A; its pull-back $f^*\mathscr{F}$ is then the locally constant sheaf on $Y_{\text{ét}}$ that corresponds to the action of $\pi_1(Y, \overline{y})$ on A obtained by composing that of $\pi_1(X, \overline{x})$ with $\pi_1(Y, \overline{y}) \to \pi_1(X, \overline{x})$.

If X is a scheme over a field k, if L is an extension of k and if $\mathscr{F}$ is an étale sheaf on X, its pull-back to X_L will be denoted by $\mathscr{F}_L$

Assume now that X is a scheme of finite type over $\mathbb{C}$ and let $\mathscr{F}$ be an étale sheaf on X. There is a natural (topological) sheaf $\mathscr{F}_{\text{top}}$ on $X(\mathbb{C})$ associated to $\mathscr{F}$. If $\mathscr{F} = A_X$ for some set A then $\mathscr{F}_{\text{top}} = A_{X(\mathbb{C})}$.

If $\mathscr{F}$ is locally isomorphic to a constant sheaf A_X, then $\mathscr{F}_{\text{top}}$ is locally isomorphic to $A_{X(\mathbb{C})}$; if one assumes moreover that A is finite and that X is connected and non-empty, there is a nice description of $\mathscr{F}_{\text{top}}$ as follows. Let $x \in X(\mathbb{C})$ and let us still denote by x the corresponding geometric point of X. The locally constant sheaf $\mathscr{F}$ is given by an an action of $\pi_1(X, x)$ on A. the sheaf $\mathscr{F}_{\text{top}}$ is then the locally constant sheaf that corresponds to the action of $\pi_1(X(\mathbb{C}), x)$ on A obtained by composing that of $\pi_1(X, x)$ with the arrow $\pi_1(X(\mathbb{C}), x) \to \pi_1(X, x)$ (remind that the right-hand term is the profinite completion of the left-hand one).

Two fundamental exact sequences

The Kummer exact sequence. Let X be a scheme and let n be an integer which is invertible on X. The *Kummer sequence* of étale sheaves on X is the sequence

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{z \mapsto z^n} \mathbb{G}_m \to 1.$$

This is an exact sequence. To see it, the only point to check is that every section of $\mathbb{G}_m$ has locally for the étale topology a preimage by $z \mapsto z^n$, that is, a n-th square root.

So, let $U \in X_{\text{ét}}$ and let f an invertible function on U. We would like to prove that there exists an étale covering (U_i) of U such that $f_{|U_i}$ is an n-th power for every i; one immediately reduces to the case where U is affine, say equal to $\mathsf{Spec}\, A$. But $\mathsf{Spec}\, A[T]/(T^n - f) \to U$ is then an étale covering (see the examples in section 4), and $\overline{T}$ is a nt-th root of f in $A[T]/(T^n - f)$.

Remark. The Kummer exact sequence is often used as an algebraic substitute for the exponential sequence.

The Artin-Schreier exact sequence. Let X be a scheme over $\mathbb{F}_p$. The map $\mathscr{O}_X \xrightarrow{z \mapsto z^p - z} \mathscr{O}_X$ is $\mathbb{F}_p$ linear, and it is not difficult to see that its kernel is precisely $\underline{\mathbb{F}_p}$. The *Artin-Schreier sequence* of étale sheaves on X is the sequence

$$0 \to \underline{\mathbb{F}_p} \to \mathscr{O}_X \xrightarrow{z \mapsto z^p - z} \mathscr{O}_X \to 0.$$

This is an exact sequence. To see it, it remains to check that every section of $\mathscr{O}_X$ has locally for the étale topology a preimage by $z \mapsto z^p - z$.

So, let $U \in X_{\text{ét}}$ and let f a function on U. We would like to prove that there exists an étale covering (U_i) of U such that $f_{|U_i}$ is equal for every i to $z_i^p - z_i$

for a suitable function z_i on U_i ; one immediately reduces to the case where U is affine, say equal to $\mathsf{Spec}\ A$. But $\mathsf{Spec}\ A[T]/(T^p - T - f) \to U$ is then an étale covering (see the examples in section 4), and $f = \overline{T}^p - \overline{T}$ in $A[T]/(T^p - T - f)$.

Étale cohomology with support

We would like to mention here two important notions related to the étale cohomology of schemes.

The cohomology with support. Let X be a scheme and let Y be a Zariski-closed subset of X. The functor that sends a sheaf of abelian groups $\mathscr{F}$ on $X_{\text{ét}}$ to the group of sections of $\mathscr{F}$ with support in Y (that is, which restrict to zero on $X - Y$) is left-exact, hence admits derived functors. The latter are denoted by $\mathrm{H}^i_Y(X_{\text{ét}}, .)$; we say that the $\mathrm{H}^i_Y(X_{\text{ét}}, \mathscr{F})$'s are the *étale cohomology groups with support in Y* of $\mathscr{F}$.

The cohomology with proper (or compact) support. Let X be a scheme, and assume that X *is separated and of finite type over a field k*. One can then define for every sheaf $\mathscr{F}$ on $X_{\text{ét}}$ the *étale cohomology groups with proper (or compact) support of $\mathscr{F}$*, which are denoted by $\mathrm{H}^\bullet_c(X_{\text{ét}}, \mathscr{F})$.

The group $\mathrm{H}^0_c(X_{\text{ét}}, \mathscr{F})$ is precisely the group of sections of $\mathscr{F}$ with proper support (that is, whose restriction to $X - Z$ is zero for some suitable closed subset Z of X which is proper), and if X is itself proper then $\mathrm{H}^i_c(X_{\text{ét}}, \mathscr{F}) = \mathrm{H}^i(X_{\text{ét}}, \mathscr{F})$ for every i and every $\mathscr{F}$, as expected.

But we would like to emphasize that in general, contrary to the definition that is used classical in topology, the $\mathrm{H}^i_c(X_{\text{ét}}, .)$'s are *not* the derived functors of $\mathrm{H}^o_c(X_{\text{ét}}, .)$, though they give rise to long exact sequences starting from short ones.

Those derived functors do exist, but are in general simply irrelevant, because there are 'not enough' closed subsets of X which are proper – if X is affine every such closed subset is a union of finitely many closed points, for example; that's why Grothendieck suggested a more *ad hoc* definition.

The definition of $\mathrm{H}^i_c(X_{\text{ét}}, \mathscr{F})$. As X is of finite type and separated over a field k, it admits a *compactification*, that is, an open immersion j into a proper k-scheme $\overline{X}$. Now define $j_!\mathscr{F}$ as the 'extension by zero' of $\mathscr{F}$, that is, the sheafification on $\overline{X}_{\text{ét}}$ of the presheaf that sends any $U \in \overline{X}_{\text{ét}}$ to $\mathscr{F}(U)$ if $U \to \overline{X}$ goes through X and to zero otherwise.

One can prove that the groups $\mathrm{H}^i(\overline{X}_{\text{ét}}, j_!\mathscr{F})$ don't depend on the chosen compactification of X, and for every i one *sets*

$$\mathrm{H}^i_c(X_{\text{ét}}, \mathscr{F}) = \mathrm{H}^i(\overline{X}_{\text{ét}}, j_!\mathscr{F}).$$

Remark. One can prove that if T is a compact topological space, if $\iota : U \hookrightarrow T$ is an open immersion, and if $\mathscr{G}$ is a sheaf on U one has natural isomorphisms

$$\mathrm{H}^i_c(U, \mathscr{G}) \simeq \mathrm{H}^i(T, \iota_!\mathscr{G})$$

where $\iota_!$ denotes the 'extension by zero' functor; this gives kind of a motivation for the above definition.

Link between étale and Zariski topology. If X is a scheme and if $\mathscr{F}$ is an étale sheaf of abelian groups on X, we know that $\mathscr{F}$ is also a sheaf for the Zariski

topology on X. It follows from the very definition of derived functors (which we didn't give in full detail) that there are for every i natural homomorphisms $\mathrm{H}^i(X_{\mathrm{Zar}}, \mathscr{F}) \to \mathrm{H}^i(X_{\mathrm{\acute{e}t}}, \mathscr{F})$; the same also holds for cohomology with support in a given Zariski-closed subset of X.

Étale cohomology of an algebraic variety over an algebraically closed field

Theorem 3. *Let k be an algebraically closed field and let X be a separated k-scheme of finite type. Let ℓ be a prime number, and let $\mathscr{F} \in \mathsf{Sh}(X, \ell)$.*

A) If $\dim X = d$ then $\mathrm{H}^i(X_{\mathrm{\acute{e}t}}, \mathscr{F}) = 0$ for $i > 2d$, and if X is affine then $\mathrm{H}^i(X_{\mathrm{\acute{e}t}}, \mathscr{F}) = 0$ for $i > d$.

B) For every i the group $\mathrm{H}^i_c(X_{\mathrm{\acute{e}t}}, \mathscr{F})$ is finite, and if L is an algebraically closed extension of k then $\mathrm{H}^i_c(X_{\mathrm{\acute{e}t}}, \mathscr{F}) \to \mathrm{H}^i_c(X_{L,\mathrm{\acute{e}t}}, \mathscr{F}_L)$ is an isomorphism.

C) Assume that X is proper or that $\ell \neq 0$ in k. Under those assumptions $\mathrm{H}^i(X_{\mathrm{\acute{e}t}}, \mathscr{F})$ is finite, and if L is an algebraically closed extension of k then $\mathrm{H}^i(X_{\mathrm{\acute{e}t}}, \mathscr{F}) \to \mathrm{H}^i(X_{L,\mathrm{\acute{e}t}}, \mathscr{F}_L)$ is an isomorphism.

D) If $k = \mathbb{C}$ one has for every i natural isomorphisms

$$\mathrm{H}^i(X_{\mathrm{\acute{e}t}}, \mathscr{F}) \simeq \mathrm{H}^i(X(\mathbb{C}), \mathscr{F}_{\mathrm{top}}) \text{ and } \mathrm{H}^i_c(X_{\mathrm{\acute{e}t}}, \mathscr{F}) \simeq \mathrm{H}^i_c(X(\mathbb{C}), \mathscr{F}_{\mathrm{top}}).$$

Remark. There is a complex-analytic analog of assertion A): if X is a complex analytic space of dimension d, then $\mathrm{H}^i(X, \mathbb{Z}) = 0$ for $i > 2d$ (because the topological dimension of X is $2d$), and if X is Stein, then $\mathrm{H}^i(X, \mathbb{Z}) = 0$ for $i > d$ (because one can prove that X has then the homotopy type of a d-dimensional CW-complex); it is a general fact that affine schemes play a role analog to that of Stein spaces.

Good behaviour. Assertion D) ensures that if one works with a scheme of finite type over $\mathbb{C}$, then the étale cohomology groups will be exactly those one expects. But this also remains true over an arbitrarily algebraically closed field, at least for some broad classes of varieties.

This claim seems probably quite vague to the reader. Let us be more precise through three examples, that of a curve, that of a torus and that of an abelian variety. We keep the notations k and ℓ and we assume that $\ell \neq 0$ in k. We fix an integer n and we set $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$

• If X is a smooth, projective, irreducible curve of genus g, then $\mathrm{H}^1(X_{\mathrm{\acute{e}t}}, \Lambda)$ is isomorphic to Λ^{2g}. If $n > 0$, if $P_1, \ldots, P_n$ are distinct k-points on X and if $U = X - \{P_1, \ldots, P_n\}$, one has an exact sequence

$$0 \to \mathrm{H}^1(X_{\mathrm{\acute{e}t}}, \Lambda) \simeq \Lambda^{2g} \to \mathrm{H}^1(U_{\mathrm{\acute{e}t}}, \Lambda) \to \bigoplus_{i=1}^n \Lambda \xrightarrow{\Sigma} \Lambda \to 0.$$

• Let $n \in \mathbb{N}$ and set $X = \mathbb{G}^n_{m,k}$. The group $\mathrm{H}^1(X_{\mathrm{\acute{e}t}}, \Lambda)$ is isomorphic to Λ^n; the connected Galois covers of X which are associated to elements of $\mathrm{H}^1(X_{\mathrm{\acute{e}t}}, \Lambda)$

are exactly the isogenies $X' \to X$ (with connected X') whose kernel embeds in Λ.

The algebra $\mathrm{H}^\bullet(X_{\text{ét}}, \Lambda)$ (multiplication is the cup-product) is isomorphic to the free exterior Λ-algebra generated by $\mathrm{H}^1(X_{\text{ét}}, \Lambda) \simeq \Lambda^n$.

• Let X be an abelian variety of dimension g over k. The group $\mathrm{H}^1(X_{\text{ét}}, \Lambda)$ is isomorphic to Λ^{2g}; the connected Galois covers of X which are associated to elements of $\mathrm{H}^1(X_{\text{ét}}, \Lambda)$ are exactly the isogenies $X' \to X$ (with connected X') whose kernel embeds in Λ.

The algebra $\mathrm{H}^\bullet(X_{\text{ét}}, \Lambda)$ is isomorphic to the free exterior Λ-algebra generated by $\mathrm{H}^1(X_{\text{ét}}, \Lambda) \simeq \Lambda^{2g}$.

Bad behaviour. We will now show that C) definitely fails if $\ell = 0$ in k and if X is non proper. For that purpose, assume that char. $k = p > 0$ and that $\ell = p$ (for the moment, we don't make any particular assumption on X). On $X_{\text{ét}}$ the Artin-Schreier exact sequence

$$0 \to \mathbb{F}_p \to \mathscr{O}_X \xrightarrow{z \mapsto z^p - z} \mathscr{O}_X \to 0$$

induces a long exact sequence

$$0 \to \mathrm{H}^0(X_{\text{ét}}, \mathbb{F}_p) \to \mathrm{H}^0(X_{\text{ét}}, \mathscr{O}_X) \to \mathrm{H}^0(X_{\text{ét}}, \mathscr{O}_X) \to \mathrm{H}^1(X_{\text{ét}}, \mathbb{F}_p) \to \mathrm{H}^1(X_{\text{ét}}, \mathscr{O}_X) \to \dots$$

Remark. Assume that X is connected and non-empty. Let $f \in \mathrm{H}^0(X_{\text{ét}}, \mathscr{O}_X)$; to its image in $\mathrm{H}^1(X_{\text{ét}}, \mathbb{F}_p)$ is associated a connected Galois cover Y of X, together with an embedding of $\mathsf{Aut}\, Y/X)$ into $\mathbb{F}_p$; let us describe concretely Y and the embedding.

- if $f = z^p - z$ for some $z \in \mathrm{H}^0(X_{\text{ét}}, \mathscr{O}_X)$ then $Y = X$ (it is the case where the image of f in $\mathrm{H}^1(X_{\text{ét}}, \mathbb{F}_p)$ is zero), and the embedding is of course the trivial one;

- if not, let Y_U denote, for $U = \mathsf{Spec}\, A$ an open affine subset of X, the scheme $\mathsf{Spec}\, A[T]/T^p - T - f$; this is an étale cover of U, and the required Galois cover $Y \to X$ is obtained by glueing the Y_U's for varying U. The embedding is then an isomorphism $\mathsf{Aut}\, Y/X \simeq \mathbb{F}_p$, defined as follows: if φ is an X-automorphism of Y, there exists a (unique) $\lambda_\varphi \in \mathbb{F}_p$ such that $\varphi^* \overline{T} = \overline{T} + \lambda_\varphi$; the assignment $\varphi \mapsto \lambda_\varphi$ is the required isomorphism.

Suppose now that $X = \mathbb{A}^1_k$. By the comparison theorem for coherent sheaves, on has $\mathrm{H}^1(X_{\text{ét}}, \mathscr{O}_X) \simeq \mathrm{H}^1(X_{\text{Zar}}, \mathscr{O}_X)$; as X is affine, the latter is zero. We thus have, in view of the above long exact sequence, a natural isomorphism

$$\mathrm{H}^1(X_{\text{ét}}, \mathbb{F}_p) \simeq k[T]/\{z^p - z, z \in k[T]\}.$$

We thus see that $\mathrm{H}^1(X_{\text{ét}}, \mathbb{F}_p)$ is infinite, and that it actually increases by extension of the algebraically closed ground field; hence both assertions of C) are false in this setting.

A few words about the proof of assertion D). Let us begin with a general remark. Let C be a site, let $\mathscr{G}$ be a sheaf on C and let U be an object of C; fix a cohomolgy class $h \in \mathrm{H}^\bullet(U, \mathscr{G})$. One can then prove that, exactly like what happens in the classical case, the class h is locally trivial on U (for the given Grothendieck topology of C, of course).

Now let us go back to assertion D). It implies, in view of the aforementioned fact, the following assertion:

(*)   if $h \in H^\bullet(X(\mathbb{C}), \mathscr{F}_{\text{top}})$, then there exists an étale covering $(X_i \to X)$ such that the pull-back of h to $X_i(\mathbb{C})$ is zero for all i.

In fact, by kind of a very general 'abstract non-sense', it turns out that to prove assertion D) (at least for cohomology without support), it is *sufficient* to prove (*). That is the way one does it when X is smooth, by using Riemann's existence theorem; the general case is proven with Hironaka's resolution of singularities.

The ℓ-adic formalism

Let us keep the notations k, X and ℓ of th. 3, and assume that $\ell \neq 0$ in k. Though they are perfectly well-defined, the groups $H^i(X_{\text{ét}}, \mathbb{Z}_\ell)$ and $H^i(X_{\text{ét}}, \mathbb{Q}_\ell)$ in the sense of derived functors are irrelevant; let us explain why through a basic example.

If X is connected an non-empty then $H^1(X_{\text{ét}}, \mathbb{Q}_\ell) \simeq \mathsf{Hom}_{\text{cont}}(\pi_1(X), \mathbb{Q}_\ell)$; as $\pi_1(X)$ is profinite and as $\mathbb{Q}_\ell$ has no torsion, the latter is zero; therefore $H^1(X_{\text{ét}}, \mathbb{Q}_\ell) = 0$, which is not reasonable (X is here *any* algebraic k-variety).

But as Grothendieck pointed out, the groups

$$\varprojlim_n H^i(X_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \text{ and } \left(\varprojlim_n H^i(X_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

are relevant. **For that reason, the notation $H^i(X_{\text{ét}}, \mathbb{Z}_\ell)$ (resp. $H^i(X_{\text{ét}}, \mathbb{Q}_\ell)$) will almost always abusively denote the first one (resp. the second one).**
The same holds for cohomology with compact support.

With those abuses of notations theorem 3 above and the three examples which follow it also hold with $\mathscr{F} = \Lambda = \mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$, the notion of a finite group having to be replaced by that of a finitely generated $\mathbb{Z}_\ell$-module or $\mathbb{Q}_\ell$-vector space (for assertion C) of th. 3 note that we suppose now $\ell \neq 0$ in k).

An interesting consequence for ℓ-adic coefficients.   Assume that $k = \mathbb{C}$ and let σ be any (that is, possibly non-continuous) automorphism of $\mathbb{C}$. The corresponding twisted variety X_σ coincides with X as a scheme; only its structure map to $\mathsf{Spec}\,\mathbb{C}$ has been changed (through σ).
Applying D) to X and X_σ and using the fact that they coincide as schemes yields to the existence of natural isomorphism of graded algebras

$$H^\bullet(X(\mathbb{C}), \mathbb{Z}_\ell) \simeq H^\bullet(X_\sigma(\mathbb{C}), \mathbb{Z}_\ell)$$

and

$$H^\bullet(X(\mathbb{C}), \mathbb{Q}_\ell) \simeq H^\bullet(X_\sigma(\mathbb{C}), \mathbb{Q}_\ell).$$

Note that the second isomorphism immediately implies that $X(\mathbb{C})$ and $X_\sigma(\mathbb{C})$ have the same Betti numbers. But when X is smooth and projective[5], one can prove it directly without using étale cohomology: Serre observed that it is a

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[5]If X is only assumed to be smooth, this is still possible using Grothendieck's GAGA results for de Rham cohomology, which involve resolution of singularities.

consequence of GAGA and of the fact that the Betti numbers of a complex projective (or, more generally, compact Kähler) manifold Y can be computed, through Hodge theory, from the dimensions of the $\mathrm{H}^q(Y, \Omega^p)$'s.

A counter-example for real coefficients. Recently, François Charles has given in [3] an example of a complex, smooth projective variety X and of an automorphism σ of $\mathbb{C}$ such that the graded *algebras* $\mathrm{H}^\bullet(X(\mathbb{C}), \mathbb{R})$ and $\mathrm{H}^\bullet(X_\sigma(\mathbb{C}), \mathbb{R})$ are not isomorphic; but note that the graded *vector spaces* $\mathrm{H}^\bullet(X(\mathbb{C}), \mathbb{R})$ and $\mathrm{H}^\bullet(X_\sigma(\mathbb{C}), \mathbb{R})$ are isomorphic, because $X(\mathbb{C})$ and $X_\sigma(\mathbb{C})$ have the same Betti numbers: the problem comes from the cup-product.

7 Étales sheaves and étale cohomology of analytic spaces

General references: [1] *and* [2] *(the latter is used only for the comparison theorem – that is, th. 5).*

We fix a non-Archimedean, complete field k.

Examples of étale sheaves

Let us begin with a remark: as open immersions are étale, any open covering of an analytic space X is an étale covering of X; therefore every sheaf on $X_{\text{ét}}$ is, when restricted to the category of open subsets of X, a usual sheaf.

We will quickly mention some important étale sheaves; as we will see, there is a great similarity between these examples and those which were considered in the scheme-theoretic context.

Constant sheaves. If A is any set, the sheaf on $X_{\text{ét}}$ which is associated to the presheaf $U \mapsto A$ sends every $U \in X_{\text{ét}}$ to the set of locally constant A-valued functions on U – which coincides with $A^{\pi_0(X)}$; as usual, it will be most of the time simply denoted by A, when there is no risk of confusion.

As for schemes, for every prime number ℓ we will denote by $\mathsf{Sh}(X, \ell)$ the category of étale sheaves on X that are locally isomorphic to a constant sheaf associated with a finite abelian ℓ-group.

Coherent sheaves. Assume for that paragraph that X is *good* (i.e. every point of X has an affinoid neighborhood) and let $\mathscr{F}$ be a coherent sheaf of $\mathscr{O}_X$-modules on X. The assignment that sends every $\pi : U \to X$ in $X_{\text{ét}}$ to $\pi^* \mathscr{F}(U)$ is a sheaf on $X_{\text{ét}}$, which is denoted by $\mathscr{F}_{\text{ét}}$ or by $\mathscr{F}$ if this doesn't cause any trouble. For that kind of sheaves, étale cohomology doesn't carry more information than the topological one: one has for every i a canonical isomorphism $\mathrm{H}^i(X_{\text{ét}}, \mathscr{F}) \simeq \mathrm{H}^i(X_{\text{top}}, \mathscr{F})$.

What happens in the non-good case ? What we have just said above remains essentially true, up to the following: coherent sheaves and their cohomology have to be considered with respect not to the usual topology of X (it is too coarse) but to the G-topology (see the definition at section 2) ; as we already mentioned, this is a Grothendieck topology, but far more set-theoretic than the étale one.

In the good case, the theories of coherent sheaves (including their cohomology groups) for the usual topology on X and for the admissible one turn out to be the same.

The multiplicative group. The assignment $U \mapsto \mathscr{O}_U(U)^*$ is a sheaf on $X_{\text{ét}}$, which is usually denoted by $\mathbb{G}_m$. If X is good, one has a natural isomorphism $\mathrm{H}^1(X_{\text{ét}}, \mathbb{G}_m) \simeq \mathrm{H}^1(X_{\text{top}}, \mathbb{G}_m)$; in the general case, as above, one has to replace the topological H^1 with the H^1 related to the G-topology on X.

The sheaf of roots of unity. For every $n \in \mathbb{N}$ the assignment

$$U \mapsto \{f \in \mathscr{O}_U(U)^*, f^n = 1\}$$

is a subsheaf of $\mathbb{G}_m$ which is denoted by μ_n. If $n \neq 0$ in k, this sheaf is locally isomorphic to the constant sheaf $\mathbb{Z}/n\mathbb{Z}$, and the *Kummer sequence*

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{z \mapsto z^n} \mathbb{G}_m \to 1$$

is exact.

Locally constant sheaves. Let A be a finite set (resp. a finite abelian group) and assume that X is connected and non-empty ; let $\overline{x}$ be a geometric point of X. As for schemes, one can establish a natural dictionary between sheaves (resp. sheaves of abelian groups) on $X_{\text{ét}}$ which are locally isomorphic to the constant sheaf A and discrete (*i.e.* with open stabilizers) actions of $\pi_{1,\mathrm{f}}(X, \overline{x})$ on A, up to conjugation by a permutation (resp. an automorphism) of A.

If A is an abelian group and if $\mathscr{F}$ is the locally constant sheaf associated with a given action of $\pi_{1,\mathrm{f}}(X, \overline{x})$ on A then $\mathrm{H}^1(X_{\text{ét}}, \mathscr{F}) \simeq \mathrm{H}^1(\pi_{1,\mathrm{f}}(X, \overline{x}), A)$; in the particular case where $\mathscr{F} = A$ this is nothing but $\mathsf{Hom}_{\text{cont}}(\pi_{1,\mathrm{f}}(X, \overline{x}), A)$, which classifies up to isomorphism the couples $(Y \to X, \iota)$ where $Y \to X$ is a connected Galois cover and ι an embedding $\mathsf{Aut}\ (Y/X) \hookrightarrow A$.

Pull-backs . If $f : Y \to X$ is a morphism of analytic spaces, to every sheaf $\mathscr{F}$ on $X_{\text{ét}}$ can be associated in a natural way a sheaf $f^*\mathscr{F}$ on $Y_{\text{ét}}$. If $\mathscr{F} = A_X$ for some set A then $f^*\mathscr{F} = A_Y$; one has $f^*\mu_{n,X} = \mu_{n,Y}$ for every n invertible on X.

If $\mathscr{F}$ is locally isomorphic to a constant sheaf A_X, then $f^*\mathscr{F}$ is locally isomorphic to A_Y (note in particular that if $\mathscr{F} \in \mathsf{Sh}(X, \ell)$ for some prime number ℓ, then $f^*\mathscr{F} \in \mathsf{Sh}(Y, \ell)$). If one assumes moreover that A is finite and that Y and X are non-empty and connected, one has a nice description of $f^*\mathscr{F}$, as follows. Let $\overline{y}$ be a geometric point of Y and let $\overline{x}$ denote the corresponding geometric point of X. The locally constant sheaf $\mathscr{F}$ is given by an action of $\pi_{1,\mathrm{f}}(X, \overline{x})$ on A; its pull-back $f^*\mathscr{F}$ is then the locally constant sheaf on $Y_{\text{ét}}$ that corresponds to the action of $\pi_{1,\mathrm{f}}(Y, \overline{y})$ on A obtained by composing that of $\pi_{1,\mathrm{f}}(X, \overline{x})$ with $\pi_{1,\mathrm{f}}(Y, \overline{y}) \to \pi_{1,\mathrm{f}}(X, \overline{x})$.

If X is a k-analytic space, if L is a complete extension of k and if $\mathscr{F}$ is an étale sheaf on X, its pull-back to X_L will be denoted by $\mathscr{F}_L$.

If $\mathscr{X}$ is a k-scheme of finite type, then to every sheaf $\mathscr{F}$ on $\mathscr{X}_{\text{ét}}$ is associated in natural way a sheaf $\mathscr{F}^{an}$ on $\mathscr{X}_{\text{ét}}^{an}$. If $\mathscr{F} = A_{\mathscr{X}}$ for some set A one has then $\mathscr{F}^{an} = A_{\mathscr{X}^{an}}$; if $\mathscr{F} = \mu_{n,\mathscr{X}}$ for some n invertible on $\mathscr{X}$ then $\mathscr{F}^{an} = \mu_{n,\mathscr{X}^{an}}$.

If $\mathscr{F}$ is locally isomorphic to a constant sheaf $A_{\mathscr{X}}$, then $\mathscr{F}^{an}$ is locally isomorphic to $A_{\mathscr{X}}^{an}$ (note in particular that if $\mathscr{F} \in \mathsf{Sh}(\mathscr{X}, \ell)$ for some prime number ℓ, then $\mathscr{F}^{an} \in \mathsf{Sh}(\mathscr{X}^{an}, \ell)$). If one assumes moreover that A is finite and that $\mathscr{X}$, and hence $\mathscr{X}^{an}$, are non-empty and connected, one has a nice description of $\mathscr{F}^{an}$, as follows. Let $\overline{x}$ be a geometric point of $\mathscr{X}^{an}$ and let $\overline{\xi}$ denote the corresponding geometric point of $\mathscr{X}$. The locally constant sheaf $\mathscr{F}$ is given by an action of $\pi_1(\mathscr{X}, \overline{\xi})$ on A; its pull-back $\mathscr{F}^{an}$ is then the locally constant sheaf on $\mathscr{X}_{\text{ét}}^{an}$ that corresponds to the action of $\pi_{1,\text{f}}(\mathscr{X}^{an}, \overline{x})$ on A obtained by composing that of $\pi_1(\mathscr{X}, \overline{\xi})$ with $\pi_{1,\text{f}}(\mathscr{X}^{an}, \overline{x}) \to \pi_1(\mathscr{X}, \overline{\xi})$.

Étale cohomology with support

As for schemes, one can define on X cohomology groups with support in a given closed subset (the definition will be the same), and also, if X is Hausdorff, cohomology groups with compact support – it will be different, and in fact simpler.

Cohomology with support. If T is a closed subset of X, the functor that sends a sheaf of abelian groups $\mathscr{F}$ on $X_{\text{ét}}$ to the group of sections of $\mathscr{F}$ with support in T (that is, which restrict to zero on $X - T$) is left-exact, hence admits derived functors. The latter are denoted by $\mathrm{H}_T^i(X_{\text{ét}}, .)$; we say that the $\mathrm{H}_T^i(X_{\text{ét}}, \mathscr{F})$'s are the *étale cohomology groups with support in T* of $\mathscr{F}$.

Cohomology with compact support. One can also define cohomology with compact support for étale sheaves on a Hausdorff analytic space. Contrary to what happens in scheme theory, it is done like in topology, that is, by deriving the functor 'global sections with compact support'.

Link between étale and usual topology. If X is an analytic space and if $\mathscr{F}$ is an étale sheaf of abelian groups on X, we know that $\mathscr{F}$ is also a sheaf for the usual topology on X. It follows from the very definition of derived functors (which we didn't give in full detail) that there are for every i natural homomorphisms $\mathrm{H}^i(X_{\text{top}}, \mathscr{F}) \to \mathrm{H}^i(X_{\text{ét}}, \mathscr{F})$; the same also holds for cohomology with support in a given closed subset of X.

Étale cohomology of analytic spaces: two general theorems

Theorem 4 (the case of an algebraically closed ground field). *Assume that k is algebraically closed and let ℓ be a prime number which is non-zero in $\widetilde{k}$. Let X be a Hausdorff k-analytic space and let $\mathscr{F} \in \mathsf{Sh}(X, \ell)$.*

1) If X is paracompact of dimension d, then $\mathrm{H}^i(X_{\text{ét}}, \mathscr{F}) = 0$ for $i > 2d$.

2) If L is an algebraically closed complete extension of k, then for every i the natural arrows $\mathrm{H}^i(X_{\text{ét}}, \mathscr{F}) \to \mathrm{H}^i(X_{L,\text{ét}}, \mathscr{F}_L)$ and $\mathrm{H}_c^i(X_{\text{ét}}, \mathscr{F}) \to \mathrm{H}_c^i(X_{L,\text{ét}}, \mathscr{F}_L)$ are isomorphisms.

3) Assume that X is a finite union of affinoid domains, each of which can be embedded in the analytification of an algebraic k-variety. The group $\mathrm{H}^i(X_{\text{ét}}, \mathscr{F})$ is finite.

Let us make some comments about the rather technical assumption on X which is made in assertion 3).

- It is automatically true as soon as the space X is compact and quasi-smooth[6] (*rig-smooth* in the rigid-analytic language).

- Compactness of X is likely sufficient for $H^i(X_{\text{ét}}, \mathscr{F})$ to be finite; but at the moment local algebraicity is technically needed for proving it – the only known way to do it being the reduction to some algebraic finiteness results (due to Deligne) through Berkovich's comparison theorem on vanishing cycles.

Theorem 5 (comparison between algebraic and analytic étale cohomology groups). *Let $\mathscr{X}$ be a separated k-scheme of finite type, let ℓ be a prime number and let $\mathscr{F} \in \mathsf{Sh}(\mathscr{X}, \ell)$.*

1) One has $H^i_c(\mathscr{X}^{an}_{\text{ét}}, \mathscr{F}^{an}) \simeq H^i_c(\mathscr{X}_{\text{ét}}, \mathscr{F})$.

2) If moreover the prime ℓ is non zero in k (but possibly zero in $\widetilde{k}$) then $H^i(\mathscr{X}^{an}_{\text{ét}}, \mathscr{F}^{an}) \simeq H^i(\mathscr{X}_{\text{ét}}, \mathscr{F})$.

About ℓ-adic formalism in analytic geometry. If ℓ is a prime number which is non zero in k then on a compact k-analytic space one can define relevant étale cohomology groups with coefficients $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$ exactly in the same way than in scheme theory. For non-compact spaces it is slightly more complicated; the only reference on that topic are, as far as we know, unpublished notes by Berkovich.

The cohomology of some analytic spaces

We will give some examples which show that the cohomology of analytic spaces over an algebraically closed field 'takes the expected values'; here it will mean that the values will be the same as they are for the analog complex analytic space. *For that section, we suppose that k is algebraically closed*, we fix a prime number ℓ which is non zero in $\widetilde{k}$, an integer n and we set $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$.

- If X in an open polydisc, then $H^i(X_{\text{ét}}, \Lambda) = 0$ for every $i > 0$.

- If X is an open poly-annulus whose dimension is denoted by d, then $H^1(X_{\text{ét}}, \Lambda) \simeq \Lambda^d$; the connected Galois covers of X which are associated to elements of $H^1(X_{\text{ét}}, \Lambda)$ are exactly the connected Kummer covers $X' \to X$ whose group embeds in Λ.

The algebra $H^\bullet(X_{\text{ét}}, \Lambda)$ is isomorphic to the free exterior Λ-algebra generated by $H^1(X_{\text{ét}}, \Lambda) \simeq \Lambda^d$.

- If X is a connected, proper, smooth analytic group of dimension g (*e.g.* the analytification of an abelian variety, or $\mathbb{G}_m^{g,an}/\mathsf{L}$ for some lattice $\mathsf{L} \subset (k^*)^g$) then $H^1(X, \Lambda) \simeq \Lambda^{2g}$; the connected Galois covers of X which are associated to elements of $H^1(X_{\text{ét}}, \Lambda)$ are exactly the isogenies $X' \to X$ (with connected X') whose kernel embeds in Λ.

The algebra $H^\bullet(X_{\text{ét}}, \Lambda)$ is isomorphic to the free exterior Λ-algebra generated by $H^1(X_{\text{ét}}, \Lambda) \simeq \Lambda^{2g}$.

8 Poincaré duality

General references: [12]*,* [13] *and* [6] *in the case of schemes;* [1] *for analytic spaces.*

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[6]This means essentially means that X can be G-locally defined by a system of equations satisfying the usual jacobian condition; it may have a boundary (*e.g.* any closed polydisc is quasi-smooth); a space is called *smooth* if it is quasi-smooth and boundaryless (*e.g* an open polydisc).

Poincaré duality theorems in both algebraic and analytic settings look formally the same; we will therefore use a common formulation.

Cup-products with supports. Let us begin with a general remark, which hold for topological spaces as well as for étale topology of schemes and analytic spaces. In those theories, cup-product pairings are also defined for cohomology with a given closed support, and for cohomology with compact support when the latter makes sense. They give rise to pairings

$$\mathrm{H}^i_Y(.,.) \times \mathrm{H}^j_Z(.,.) \to \mathrm{H}^{i+j}_{Y \cap Z}(.,.)$$

and

$$\mathrm{H}^i_c(.,.) \times \mathrm{H}^j(.,.) \to \mathrm{H}^{i+j}_c(.,.)$$

which behave as expected.

Poincaré duality: context and notations. Now, let us denote by k an algebraically closed field (resp. an algebraically closed non-Archimedean field). Let X be a separated, smooth, k-scheme of finite type (resp. an Hausdorff smooth k-analytic space) of pure dimension d. We fix a prime number ℓ which is non zero in k (resp. $\widetilde{k}$) and an integer n; set $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$.

If $\mathscr{F} \in \mathsf{Sh}(X, \ell)$ is annihilated by Λ and if $i \in \mathbb{N}$, we will denote by $\mathscr{F}(i)$ the sheaf $\mathscr{F} \otimes \mu_{\ell^n}^{\otimes i}$, and by $\mathscr{F}^\vee$ the sheaf that sends U to $\mathrm{Hom}(\mathscr{F}_{|U}, \Lambda_U)$ (it is called the *dual* sheaf of $\mathscr{F}$); if $i \in \mathbb{Z} - \mathbb{N}$ the sheaf $\mathscr{F}(-i)^\vee$ will be denoted by $\mathscr{F}(i)$. The formulas $\mathscr{F}(i+j) = \mathscr{F}(i) \otimes \mathscr{F}(j)$ and $\mathscr{F}(i)^\vee = \mathscr{F}(-i)$ then hold for all i, j in $\mathbb{Z}$.

Theorem 5 (Poincaré duality for schemes and analytic spaces). *We use the notations above.*

1) There is a natural trace *map* $\mathsf{Tr} : \mathrm{H}^{2d}_c(X_{\text{ét}}, \Lambda(d)) \to \Lambda$.

2) For every $\mathscr{F} \in \mathsf{Sh}(X, \ell)$ which is annihilated by Λ and every $i \in \{0, \ldots, 2d\}$ the map

$$\mathrm{H}^i(X_{\text{ét}}, \mathscr{F}) \times \mathrm{H}^{2d-i}_c(X_{\text{ét}}, \mathscr{F}^\vee(d)) \to \Lambda,$$

induced by the cup-product pairing and the trace map, identifies $\mathrm{H}^i(X_{\text{ét}}, \mathscr{F})$ with the dual as Λ-module of $\mathrm{H}^{2d-i}_c(X_{\text{ét}}, \mathscr{F}^\vee(d))$, that is, with

$$\mathsf{Hom}_\Lambda(\mathrm{H}^{2d-i}_c(X_{\text{ét}}, \mathscr{F}^\vee(d)), \Lambda).$$

Let us make some comments.

A particular fundamental case. If X is connected and non-empty, then $\mathrm{H}^0(X_{\text{ét}}, \Lambda) \simeq \Lambda$; we thus deduce of 2) that the trace map induces an isomorphism $\mathrm{H}^{2d}_c(X_{\text{ét}}, \Lambda(d)) \simeq \Lambda$.

About duality. In the algebraic case, duality assertion 2) only involves *finite* Λ-modules. But that is not the case in the analytic context. Let us give a very simple example of that phenomenon in the *zero-dimensional* case (which is, maybe surprisingly, not completely empty): if X is a disjoint union of one-point spaces $\mathscr{M}(k)$ indexed by some set I (possibly infinite) then $\mathrm{H}^0(X, \Lambda) = \Lambda^I$ and $\mathrm{H}^0_c(X, \Lambda) = \Lambda^{(I)}$; classical duality between those two groups is then a particular case of Poincaré duality !

About comparison with transcendent Poincaré duality. In general topology, Poincaré duality between the cohomology without support and that with compact support on a purely-dimensional topological variety involves the *orientation sheaf* of the variety (which is locally isomorphic to the constant sheaf $\mathbb{Z}$) ; in étale cohomology, the role of the latter is, in view of the theorem above, played by the sheaf of roots of unity.

But while working with smooth complex analytic spaces, the orientation sheaf is usually non mentioned: it is because every smooth complex analytic space X comes with a natural orientation (defined through multiplication by i), whence an isomorphism between the orientation sheaf of X and $\mathbb{Z}_X$. But we emphasize that this identification is related to the choice of one of the two square-roots of -1 in $\mathbb{C}$; such a choice also induces for every N an isomorphism $\mu_N \simeq \mathbb{Z}/N\mathbb{Z}$ (by distinguishing $e^{2i\pi/N}$), modulo which the isomorphisms of th. 3, assertion D) are compatible with algebraic and transcendent Poincaré dualities for N-torsion finite coefficients.

About the definition of the trace map. We keep the notations of the theorem above. Let D be an irreducible and reduced divisor on X, that is, a reduced closed subscheme (resp. analytic subspace) of dimension $d-1$. As X is smooth, D is Cartier (*i.e.* locally defined by one non-zero equation). The sheaf of rational (or meromorphic) functions defined on $X - D$ and having at most simple poles along D is then a line bundle $\mathscr{O}(D)$ with a canonical section 1, which generates $\mathscr{O}(D)$ on $X - D$. To such a datum is associated in a quite formal way a cohomology class of $\mathrm{H}^1_D(X_{\mathrm{top}}, \mathbb{G}_m)$ which one can push to $\mathrm{H}^1_D(X_{\mathrm{ét}}, \mathbb{G}_m)$, and then to $\mathrm{H}^2_D(X_{\mathrm{ét}}, \mu_{\ell^n}) = \mathrm{H}^2_D(X_{\mathrm{ét}}, \Lambda(1))$ through the long-exact cohomology sequence associated with the Kummer exact sequence of étale sheaves

$$1 \to \mu_{\ell^n} \to \mathbb{G}_m \xrightarrow{z \mapsto z^{\ell^n}} \mathbb{G}_m \to 1.$$

This construction extends to an assignment which sends every Zariski closed subset (resp. every closed analytic subset) Y of X of pure codimension i to a cohomology class in $\mathrm{H}^{2i}_Y(X_{\mathrm{ét}}, \Lambda(i))$, which can be pushed to $\mathrm{H}^{2i}_c(X_{\mathrm{ét}}, \Lambda(i))$ if Y is proper. The formation of those cohomology classes associated with Zariski-closed (resp. closed analytic) subsets of X makes *transverse* intersections commute with cup-products.

Now let x be a k-point on X. By the above, one can associate to it a class $h(x) \in \mathrm{H}^{2d}_c(X_{\mathrm{ét}}, \Lambda(d))$. One then has $\mathsf{Tr}(h(x)) = 1 \in \Lambda$. As a consequence, if X is non-empty and connected, then $h(x)$ doesn't depend on x and generates the free Λ-module $\mathrm{H}^{2d}_c(X_{\mathrm{ét}}, \Lambda(d))$; it is called the *fundamental class* of X.

9 The case of a non-algebraically closed ground field

General references: [11] *or* [6] *for algebraic varieties, and* [1] *for analytic spaces.*

Up to now most of our theorems about the cohomology of algebraic varieties or analytic spaces concerned the case of an *algebraically closed* field. We would like to give some explanations about what happens if one removes that assumption.

Étale cohomology of a field and Galois cohomology

Let k be a field, let k^s be a separable closure of k and let G be the profinite group $\mathsf{Gal}\,(k^s/k)$. If $\mathscr{F}$ is an étale sheaf on $\mathsf{Spec}\,k$ and if L is a finite sub-extension of k^s/k, then $\mathsf{Spec}\,L \to \mathsf{Spec}\,k$ is étale; therefore $\mathscr{F}(\mathsf{Spec}\,L)$ does make sense, and will be written $\mathscr{F}(L)$ for the sake of simplicity. The direct limit of the $\mathscr{F}(L)$'s, for L going through the set of finite sub-extensions of k^s/k, is in a natural way a discrete G-set (discrete means that the stabilizers are open); we denote it by $\mathscr{F}(k^s)$; if $\mathscr{F}$ is a sheaf in abelian groups, then $\mathscr{F}(k^s)$ is a discrete G-module.

Theorem 6. *We use the above notations.*

1) The assignment $\mathscr{F} \mapsto \mathscr{F}(k^s)$ induces an equivalence *(i.e. a categorical 1-1 correspondence) between the category of étale sheaves on* $\mathsf{Spec}\,k$ *and that of discrete G-sets.*

2) If $\mathscr{F}$ is an étale sheaf on $\mathsf{Spec}\,k$ and if L is a finite sub-extension of k^s/k corresponding to an open subgroup H of G then $\mathscr{F}(L) = \mathscr{F}(k^s)^H$.

3) If $\mathscr{F}$ is a sheaf of abelian groups on $\mathsf{Spec}\,k$ then one has for every i an isomorphism $\mathrm{H}^i((\mathsf{Spec}\,k)_{\text{ét}}, \mathscr{F}) \simeq \mathrm{H}^i(G, \mathscr{F}(k^s))$.

Some comments. We therefore see that the étale cohomology theory of $\mathsf{Spec}\,k$ is nothing but the cohomology theory of discrete $\mathsf{Gal}\,(k^s/k)$-modules, which is also called the *Galois cohomology* of the field k. Galois cohomology plays a crucial role in modern arithmetic – we suggest the interested reader to look at Serre's book ([8]).

A topological analog. Let X be a connected, non-empty, 'reasonable' topological space, admitting a universal cover $Y \to X$; let G be the automorphism group of Y over X. Let A be an abelian group equipped with an action of G. It corresponds to a local system $\mathscr{F}$ on X which is locally isomorphic to A; the pull-back of $\mathscr{F}$ on Y is *globally* isomorphic to A. The group G acts in a natural way on $\mathrm{H}^\bullet(Y, A)$.

There is a natural spectral sequence

$$\mathrm{H}^p(G, \mathrm{H}^q(Y, A)) \Rightarrow \mathrm{H}^{p+q}(X, \mathscr{F}).$$

Now assume that $\mathrm{H}^q(Y, A) = 0$ for every $q > 0$ (*e.g* Y is contractible). Then this spectral sequences simply furnishes isomorphisms $\mathrm{H}^p(X, \mathscr{F}) \simeq \mathrm{H}^p(G, A)$ for every p. Therefore we see that, as far as étale cohomology is concerned, the spectrum of a field k behaves, to some extent, like a connected, non-empty topological space having a contractible universal cover; the spectrum of a separable closure k^s of k can be thought as such a cover.

Varieties or analytic spaces over a field

Our purpose is now to explain how the cohomology of a variety or an analytic space over a field k mixes, in some sense, Galois cohomology of the ground field and cohomology of the given variety over an algebraic closure of k.

Let us begin with some remarks. Fix a field k and an algebraic closure k^a of k ; let G be the absolute Galois group $\mathsf{Gal}\,(k^a/k)$.

1) If X is an algebraic variety over k and if $\mathscr{F}$ is an an étale sheaf of abelian groups on X, there is a natural discrete action of G on $\mathrm{H}^\bullet(X_{k^a, \text{ét}}, \mathscr{F}_{k^a})$.

2) If k is non-Archimedean and complete, and if $\widehat{k^a}$ denotes the completion of k^a, then for every *compact*[7] analytic space X and every étale sheaf of abelian groups $\mathscr{F}$ on X, there is a natural discrete action of G on $\mathrm{H}^\bullet(X_{\widehat{k^a},\text{ét}}, \mathscr{F}_{\widehat{k^a}})$.

Now we will see that in the first (resp. second) case, the cohomology of $\mathscr{F}$ is builded from the $\mathrm{H}^p(G, \mathrm{H}^q(X_{k^a,\text{ét}}, \mathscr{F}_{k^a}))$'s (resp. $\mathrm{H}^p(G, \mathrm{H}^q(X_{\widehat{k^a},\text{ét}}, \mathscr{F}_{\widehat{k^a}})$'s) through a spectral sequence.

Let us now state a precise theorem.

Theorem 7. *Let k be a field and let k^a be an algebraic closure of k; let G be the absolute Galois group* $\mathsf{Gal}\,(k^a/k)$.

1) If X is a k-scheme of finite type and if $\mathscr{F}$ is an étale sheaf of abelian groups on X, one has a natural spectral sequence

$$\mathrm{H}^p(G, \mathrm{H}^q(X_{k^a,\text{ét}}, \mathscr{F}_{k^a})) \Rightarrow \mathrm{H}^{p+q}(X_{\text{ét}}, \mathscr{F}).$$

2) Assume that k is non-Archimedean and complete, and let $\widehat{k^a}$ be the completion of k^a. If X is a compact *analytic space over k and if $\mathscr{F}$ is an étale sheaf of abelian groups on X, one has a natural spectral sequence*

$$\mathrm{H}^p(G, \mathrm{H}^q(X_{\widehat{k^a},\text{ét}}, \mathscr{F}_{\widehat{k^a}})) \Rightarrow \mathrm{H}^{p+q}(X_{\text{ét}}, \mathscr{F}).$$

References

[1] V. Berkovich, *Étale cohomology for non-archimedean analytic spaces*, Inst. Hautes Etudes Sci. Publ. Math. **78** (1993) 5-161.

[2] , *On the comparison theorem for étale cohomology of non-Archimedean analytic spaces*, Israel J. Math. **92** (1995), 45-60.

[3] F. Charles, *Conjugate varieties with distinct real cohomology algebras*, J. Reine Angew. Math. **630** (2009), pp. 125–139.

[4] A. Grothendieck, *Sur quelques points d'algbre homologique*, Tohoku Mathematics Journal **9**, 119-221.

[5] L. Illusie, *Grothendieck et la topologie étale*, available on Luc Illusie's home page.

[6] J. S. Milne, *Étale cohomology*, Princeton university press, Princeton, NJ (1980).

[7] J.-P. Serre, *Exemple de varits projectives conjugues non homomorphes*, C. R. Acad. Sci. Paris **258** (1964), pp. 4194-4196.

[8] J.-P. Serre, *Cohomologie galoisienne*, 5th edition, Lecture note in mathematics **5**, Springer-Verlag 1997.

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[7]The fact for the action of G on the cohomology of $\mathscr{F}_{\widehat{k^a}}$ to be discrete essentially means that every cohomology class of $\mathscr{F}_{\widehat{k^a}}$ is already defined on a finite separable extension of k; one needs compactness to ensure this finiteness result.

[9] SGA 1, *Revêtements étales et groupe fondamental*, Lecture note in mathematics **224**, Springer-Verlag 1971.

[10] SGA 4-1, *Théorie des topos et cohomologie étale des schmas*, volume 1, Lecture note in mathematics **269**, Springer-Verlag (1972).

[11] SGA 4-2, *Théorie des topos et cohomologie étale des schmas*, volume 2, Lecture note in mathematics **270**, Springer-Verlag (1972).

[12] SGA 4-3, *Théorie des topos et cohomologie étale des schmas*, volume 3, Lecture note in mathematics **305**, Springer-Verlag (1972).

[13] SGA $4\frac{1}{2}$, *Cohomologie étale*, Lecture note in mathematics **569**, Springer-Verlag (1977).

[14] SGA5, *Cohomologie ℓ-adique et Fonctions L*, Lecture note in mathematics **589**, Springer-Verlag (1977).